Problem Set 2 Solutions

Due: September 17, 2015

1 Starred Problem 1

\( ty' - (t + 2)y = 3t^4 \) divide by \( t \) to get into standard form

\( y' - \frac{t + 2}{t} = 3t^3 \) use integrating factor

\[ \mu(t) = \exp \left( \int \frac{1}{t} \, dt \right) = \exp(-t - 2 \ln(t)) = t^{-2}e^{-t} \]

\[ \mu y = \int \mu 3t^3 \, dt = \int 3te^{-t} \, dt \text{ now integrate by parts} \]

\[ t^{-2}e^{-t}y = -3(t + 1)e^{-t} + c \]

\[ y = -3t^2(t + 1) + ct^2e^t \text{ now use inition condition } y(1)=1 \text{ to find } c \]

\[ 1 = -6 + 7e^{-1} \]

\[ y(t) = -3t^2(t + 1) + 7t^2e^{t-1} \text{ or equivalent} \]

2 Starred Problem 2

\[ y' = \frac{(y - 1)(y + 2)}{t + 1} \text{ separate the equations} \]

\[ \frac{dy}{(y - 1)(y + 2)} = \frac{dt}{t + 1} \text{ integrate both sides, use partial fractions for } y \]

\[ \ln(y - 1) - \ln(y + 2) = 3 \ln(t + 1) + c \text{ manipulate to get } y \text{ alone} \]

\[ \ln \left( \frac{y - 1}{y + 2} \right) = \ln \left( (t + 1)^3 \right) + c \]

\[ \frac{y - 1}{y + 2} = A(t + 1)^3 \text{ use initial condition to find } A \]

\[ \frac{-0.5 - 1}{-0.5 + 2} = A(0 + 1)^3 \implies A = -1 \]

\[ y - 1 = A(t + 1)^3(y + 2) \]

\[ y - 1 = A(t + 1)^3y + 2A(t + 1)^3 \]
\[ y - A(t+1)^3 y = 2A(t+1)^3 + 1 \]  
\[ y = \frac{2A(t+1)^3 + 1}{1 - A(t+1)^3} \]  
\[ y(t) = \frac{1 - 2(t+1)^3}{(t+1)^3 + 1} \]

Note: A maximum of 7/10 points shall be awarded for solutions that do not solve for \( y \) as a function of \( t \).

3 Starred Problem 3

\( V_{in} = V_{out} \), so we have volume in the tank is a constant \( V = 100 \, gal \) Initially there is 40 lbs of salt in the tank, \( Q_0 = 40 \, lbs \) \( C_{in} = 2 \, lb/gal \); \( V_{in} = 5 \, gal/min \); \( C_{out} = \) concentration of the tank = \( Q/V \), \( V_{out} = 5 \, gal/min \)

\[ Q' = \text{rate in - rate out} \]  
\[ Q' = C_{in}V_{in} - C_{out}V_{out} \]  
\[ Q' = 10lb/min - Q/100 \ast 5/min \] Q has units lb, so all units make sense  
\[ Q' = 10 - \frac{Q}{20} \text{ ignoring units for now, this will be our DE that models the problem at hand} \]

Now that we have a DE, we need to solve it. Remember the initial condition \( Q_0 = 40 \, lbs \). We can solve it using chapter 1 methods or integrating factors as it is a 1st order linear equation with constant coefficients.

\[ Q' = 10 - \frac{Q}{20} \]  
\[ \frac{dQ}{10 - \frac{Q}{20}} = dt \]  
\[ -20 \ln(10 - Q/20) = t + c \]  
\[ 10 - Q/20 = Ae^{-t/20} \]  
\[ Q = 20(10 - Ae^{-t/20}) \]  
\[ Q(t) = 20(10 - 8e^{-t/20}) \, lbs \]

Note: initial concentration is 40lb/100gal = .4 lb/gal, incoming concentration is 2 lb/gal. Thus we expect the concentration in the tank to increase and thus the mass to increase so our solution makes sense physically.  
Also no points shall be deducted for ignoring units.