Numerical Computing
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Problem Set 7

Due:
Tuesday, 11/17/98

1. The trapezoidal rule, including remainder term, is

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\,(f(a)+f(b)) + C(b-a)^{p+2} f^{(p+1)}(\mu)$$

where p is the precision, C is a constant, and $\mu \in [a,b]$.

(a) Consider the monomials $f(x) = x^k$, $k = 0, 1, \ldots$ to verify that the precision of the trapezoidal rule is 1.

(b) Find C.

2. Let $I = \displaystyle\int_0^3 \sqrt{1+x^3}\,dx$.

(a) Approximate I using the composite Simpson rule with 3 subintervals.

(b) Approximate I using the three-point Gaussian quadrature formula.

3. Consider the finite difference formula

$$f'(0) = af(0) + bf(h) + cf(2h) + E(f)$$

where h, a, b, and c are constants and $E(f)$ is the error (for a given function f).

(a) Determine a, b, and c by requiring that $E(f) = 0$ when $f(x) = 1$, x, and x^2.

(b) Suppose E has the form $Kf'''(\mu)$, where K is a constant and $\mu \in [0, 2h]$. Find K using $f(x) = x^3$.

4. It is known that

$$f'(0) = \frac{f(h) - f(-h)}{2h} + C_1 h^2 + C_2 h^4 + \ldots$$

where C_1 and C_2 are constants independent of h. Derive an $O(h^4)$ accurate formula for $f'(0)$ using Richardson extrapolation.

5. Let $I = \int_0^\pi \sin(x)\,dx = 2$. The object of this exercise is to obtain high order accurate approximations to this integral using a method known as Romberg integration. Using Matlab, enter the following commands:

```
n=5; R=zeros(n,n);
for i=1:n
x=linspace(0,pi,2∧(i-1)+1);
y=sin(x);
R(i,1)=trapz(x,y);
end
```

The first column of R contains composite trapezoidal rule approximations of I with increasing numbers of subintervals. Print the first column of R (using `format long`). How does the error in $R(i,1)$ behave as a function of i?

Higher order approximations can be obtained from the entries in the first column using Richardson extrapolation. Enter the commands

```
for i=2:n
for j=2:i
R(i,j)=(4∧(j-1)*R(i,j-1)-R(i-1,j-1))/(4∧(j-1)-1);
end
end
```

Now print out the lower triangular matrix R (again using `format long`). Characterize the behavior of the error in each column of R as a function of i.