Note: No solutions for these problems will be published. You are strongly encouraged to try working on the problems ALONE before working with others. This is the only way you can realistically gauge how prepared you are for an exam atmosphere. We will be able to discuss these problems during the review class: see my webpage for time and location.

NOTE: I didn't include too many problems from the exam #3 material since we just discussed that recently. This doesn't mean that the exam #3 material is less important.

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1. Solve the initial value problem:

$$y' + t^2 y = t^2, \qquad y(0) = 2.$$

2. Consider the initial value problem:

$$y' + \frac{1}{2x} y = x^{3/2}, \qquad y(1) = a$$

a.) What's the largest interval on which we're guaranteed a unique solution to this initial value problem?

b.) Find the solution to the initial value problem in terms of a.

c.) Does the limiting behavior of the solution as $x \to \infty$ depend on the value of a? Explain.

3. Find an explicit solution to the initial value problem:

$$\frac{dy}{dx} = 2xy^2 + 2y^2, \qquad y(0) = \frac{1}{3}.$$

On what interval is this solution guaranteed to be valid?

4. In attempting to model the behavior of a population of geese, you come up with the autonomous equation: $y' = -y(y^2 - 9y + 18)$ where y is in thousands of geese.

a.) Find all critical points for this equation.

b.) By sketching the graph of y' vs. y, classify each critical point as stable or unstable.

c.) Sketch qualitatively accurate solutions corresponding to each of the following initial populations:

$$y(0) = 1 \quad y(0) = 3 \quad y(0) = 5 \quad y(0) = 10$$

5. A bartender in Hawaii has two kinds of Papaya juice: type A is 100 % pure; type B is only 80 % papaya juice in a water solution. The bartender believes the ideal concentration of papaya juice to be 92 %, and resolves to mix up a batch with this concentration.

He starts with a tank containing 10 liters of the pure type A, and begins pumping in type B at a rate of 1 liter per hour, while pumping the well-mixed juice out at the same rate.

a.) Set up an initial value problem for $Q(t)$, the amount (in liters) of pure papaya juice in the tank as a function of time. (For example, if at time t_0 the concentration of the juice in the tank is 96 %, then $Q(t_0) = 9.6$.)

b.) Solve this initial value problem for $Q(t)$.

c.) If you wanted to determine the time after which the concentration of juice in the tank was 92 %, what equation would you have to solve?

6. Consider the non-homogeneous equation:

$$y'' - 4y' + 3y = e^t + 5e^{2t}.$$

a.) Find the homogeneous solution.

b.) Find the general solution.

c.) In solving the equation $y'' + 3y' - 4y = t^2 e^{2t}$, what form should be assumed for a particular solution?

7. A mass weighing 288 lbs. stretches a spring 18 feet at equilibrium. The mass is displaced to 2 feet below the equilibrium position and released with a downward velocity of 8/3 feet per second.

a.) Find the natural frequency and natural period.

b.) If the system is undamped, find and sketch the solution $u(t)$.

c.) Give an example of a value of the damping coefficient γ that will make the system underdamped. Sketch the solution for this value of γ.

8. Consider the function

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 1, & 1 < x < 2 \end{cases}$$

a.) Sketch, on the interval $-4 \leq x \leq 4$, the graph of $f_e(x)$, the even periodic extension of $f(x)$ which has period $T = 2L = 4$.

b.) If the Fourier Cosine Series for $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2}),$$

find $\frac{a_0}{2}$ and an expression for $a_n, n \geq 1$. Simplify this expression as much as possible.

c.) Sketch the graph of the function to which the above Fourier series converges.

9. Consider the Euler equation

$$x^2 y'' + 3xy' - 8y = 0, \qquad x > 0$$

a.) By assuming solutions of the form $y = x^r$, find the general solution.

b.) Suppose we didn't know how to solve the equation as in part a.), but were given that one solution is $y_1(x) = x^2$. Use Abel's Theorem to find a second independent solution $y_2(x)$.

10. Consider the 2 x 2 system:

$$\mathbf{x}' = \begin{pmatrix} -1 & \gamma \\ 2 & -4 \end{pmatrix} \mathbf{x}.$$

a.) Suppose $\gamma = -1$. Find the eigenvalues of the coefficient matrix, find the general solution, and sketch a phase portrait.

b.) Assuming that γ is real, discuss the different possibilities for the type and stability of the critical point at the origin. Make sure to discuss what values of γ give rise to each type of behavior.