Math-6500 Partial Differential Equations Fall 2007

The Laplace Equation

1. Ill-posedness of the Cauchy problem for Laplace's equation (*cf. Guenther & Lee* Sec. 1-8). In two dimensions consider the problem

   $$\Delta u = 0,\ \infty < x < -\infty,\ y > 0,$$
   $$u(x,0) = f(x),\ u_y(x,0) = g(x),\ \infty < x < -\infty$$

   Construct a sequence of separated solutions

   $$u_n(x,y) = \frac{1}{n} Y_n(y) \cos nx$$

   such that $u_n(x,0) \to 0,\ (u_n)_y(x,0) \to 0$ as $n \to \infty$, while $u_n(x,1) \to \infty$. This example shows that the solution doesn't depend continuously on the initial data: indeed, the data converge

   $$F_n(x) = f(x) + u_n(x,0) \to f(x),\ G_n(x) = g(x) + (u_n)_y(x,0) \to g(x)$$

   but the corresponding solutions do not converge $u(x,y) + u_n(x,y) \nrightarrow u(x,y)$.

2. (Based on Problem 4.1 #2 of *Partial Differential Equations* by F. John). Let $L = \Delta + c$ in $n = 3$ dimensions, where $c > 0$ is a constant. (L is Helmholtz or reduced wave operator).

   (a) Find all solutions of $Lu = 0$ with spherical symmetry. Hint: Set $u(r) = v(r)/r$.

   (b) Prove that

   $$\Phi(x) = -\frac{\cos(\sqrt{c}\,|x|)}{4\pi\,|x|}$$

   is a fundamental solution for L. In other words,

   1. Use the divergence theorem to prove that

      $$\int_{|x|<a} L\Phi\,dx = 1$$

   2. * Repeat the argument from *Evans* (paragraphs 2–4, pages 24-25) to demonstrate that

      $$u(x) = \int_{\mathbb{R}^3} \Phi(x-y) f(y)\,dy$$

      solves $Lu = f$.

Distributions $\mathcal{D}'$

3. Prove that $\lim_{n\to\infty} \sin(nx) = 0$ in the space of distributions $\mathcal{D}'(\mathbb{R})$, i.e., that

$$\lim_{n\to\infty} \int \sin(nx)\phi(x)\,dx = 0$$

for any $\phi \in \mathcal{D} = C_0^\infty$; this result is a simple form of the Riemann-Lebesgue lemma of harmonic analysis. Find $\lim_{n\to\infty} \sin^2(nx)$ in the space of distributions $\mathcal{D}'(\mathbb{R})$.

Comment: This problem shows that multiplication of distributions is not continuous, even when it is defined.

4. (a) Prove that $x\delta(x) = 0$ in the sense of distributions. Thus, in $\mathcal{D}'(\mathbb{R})$ the equation $xT(x) = 0$ has a solution $T(x) = c\delta(x)$ (cf. *Guenther & Lee* Problem 10-5.4*)*.

   (b) Solve the differential equation $dT/dx = \delta(x)$ (the solution then is a fundamental solution of the equation). Don't forget to prove that your solution satisfies the equation in $\mathcal{D}'(\mathbb{R})$.

5. Prove that the function $G(x_1, x_2)$ defined by

$$G(x_1, x_2) = \begin{cases} 1, & \text{for } x_1 > \xi_1, x_2 > \xi_2 \\ 0, & \text{elswere} \end{cases}$$

is a fundamental solution with the pole (ξ_1, ξ_2) of the operator $L = \partial^2/\partial x_1 \partial x_2$ in the x_1x_2-plane, so that in the sense of distributions $LG = \delta(x - \xi)$. In other words, for each test function $\phi \in \mathcal{D} = C_0^\infty(\mathbb{R}^2)$,prove that

$$\int G\ L\phi\ dx = \phi(\xi_1, \xi_2)$$