

## A FINITE-DIMENSIONAL ATTRACTOR FOR A NONEQUILIBRIUM STEFAN PROBLEM WITH HEAT LOSSES

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**Abstract.** We study a two-phase modified Stefan problem modeling solid combustion and nonequilibrium phase transitions. The problem is known to exhibit a variety of non-trivial dynamical scenarios. In the presence of heat losses we develop a priori estimates and establish well-posedness of the problem in weighted spaces of continuous functions. The estimates secure sufficient decay of solutions that allows us to carry out analysis in Hilbert spaces. We demonstrate existence of a compact attractor in the weighted spaces and prove that the attractor consists of sufficiently regular functions. We show that the Hausdorff dimension of the attractor is finite.

**1. Introduction.** The subject of this paper is a study of dynamics of a nonequilibrium two-phase Stefan problem modeling condensed phase combustion and some phase transition processes. It was demonstrated numerically [9] that the sharp interface model of the condensed phase combustion also known as Self-propagating High-temperature Synthesis (SHS) generates a remarkable variety of complex thermokinetic oscillations. In addition to its theoretical interest SHS finds technological applications as a method of synthesizing certain technologically advanced materials, see [21], [26] and also [27] for a popular exposition. The process is characterized by highly exothermic reactions propagating through mixtures of fine elemental reactant powders, resulting in the synthesis of compounds. The propagation can be thought of as a traveling wave, whose velocity and shape change during the evolution. The best representation of dynamical changes in the wave shape are clearly obtained in the coordinate frame traveling with the velocity of the wave. Depending on parameters of the model, dynamical scenarios exhibited by the model include a Hopf bifurcation, period doubling cascades leading to chaotic pulsations, a Shilnikov-Hopf bifurcation etc. These scenarios are well-known for the finite-dimensional dynamical systems and suggest a possibility that the essential dynamics of the free-interface problem may be finite-dimensional as well. Indeed, we have been able to prove [12, 13] that compactness and finite dimensionality of the attractor take place for a simpler one-phase problem.

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However, the methods of the papers dealing with the one-phase problem are not directly applicable to the sharp interface problem of the condensed phase combustion which is the subject of the present paper. The principal difficulty that arises here as compared to the one-phase problem is that the presence of the additional temperature field behind the propagating interface (in the product phase) creates an additional degree of freedom that is not easily controllable. This difficulty is overcome in the present paper; *we show that Hausdorff dimension of the attractor is finite*. The paper draws on the approach of our previous work [14] discussing compactness of the attractor; in addition to the dimension estimate, results presented in this paper clarify structure and regularity properties of the attractor.

There is a substantial literature that treats analytical aspects of the initial-boundary value problem for different sharp-interface models with kinetics, related to the problem (1.1-1.4) below, see [19, 22, 3, 6]. These works are concerned with basic issues of mostly local in time existence. We also note recent papers by Brauner *et al.* and Lorenzi, [1, 2, 18], which study weakly-nonlinear dynamical behavior of solutions of related problems. In particular they consider perturbations of traveling-wave initial data and investigate their instability and bifurcations. In contrast, the principal focus of the present paper is in strongly nonlinear asymptotic dynamics for a wide range of initial data and parametric regimes.

The free-interface problem is formulated as follows: find  $s(t)$  and  $u(x, t)$  such that

$$u_t = u_{xx} - \gamma u, \quad x \neq s(t), \quad t > 0, \quad (1.1)$$

$$u(x, 0) = u_0(x) \geq 0, \quad (1.2)$$

$$g[u(s(t), t)] = v(t), \quad (1.3)$$

$$[u_x(s(t), t)] := u_x^+(s(t), t) - u_x^-(s(t), t) = v(t) \quad (1.4)$$

where  $v(t)$  is the interface velocity,  $s(t) = \int_0^t v(\tau) d\tau$  is its position,  $u$  is the temperature, and the derivatives  $u_x^+$  and  $u_x^-$  are taken from right side and left side of the free interface respectively. The last term in the heat equation (1.1) is due to the heat losses into the medium surrounding the combustible or solidifying substance via Newton's cooling law with a non-dimensional coefficient  $\gamma > 0$ .

Dynamics of the physical system is determined by the feedback mechanism between the heat release due to the kinetics  $g(u|_{x=s(t)})$  and the heat dissipation by the medium. The second interface condition (1.4) (the Stefan boundary condition) expresses the balance between the heat produced at the free boundary and its diffusion by the adjacent medium. As the problem describes, generally speaking, propagation of a phase transition front, the first interface condition (1.3) is a manifestation of the *nonequilibrium* nature of the transition; its analog for the classical Stefan problem is just  $u|_{x=s(t)} = 0$ . We should mention that in contrast with the nonequilibrium problem, the dynamics of the classical Stefan problem is relatively trivial. The surrounding matter is assumed to be at the temperature of the fresh combustible mixture at  $-\infty$  (the original phase in the phase transition interpretation). By the same token the heat loss will reduce the temperature in the product phase to that of the medium. Thus the behavior of the solution at infinity should satisfy  $\lim_{x \rightarrow \pm\infty} u(x, t) = 0$ .

In order to estimate the Hausdorff dimension of the attractor we need to develop some additional technical tools. We first develop a priori estimates and establish well-posedness of the problem in weighted spaces of continuous functions, which are introduced for the statement of the problem in the coordinate frame moving

with the interface (i.e., where the interface is fixed at zero). These estimates, which constitute the analytical core of the paper, secure sufficient decay of solutions that allows us to carry out analysis in a Hilbert space.

It should be noted that volume estimates, which are the basis for the Hausdorff dimension bound, require a Hilbert structure in the underlying space. Next we are able to extend our results on existence of compact attractors [14] to the weighted spaces. After that we study the evolution on the attractor and prove that the semigroup on the attractor is onto and one-to-one: it yields that the attractor consists of sufficiently regular functions. As a consequence we are able to demonstrate that problem is well-posed and its attractor is precompact in a Hilbert space.

This allows us to estimate the Hausdorff dimension of the attractor based on the techniques described for instance in [24]. We study evolution of the infinitesimal volume along the trajectories in the attractor and demonstrate that for sufficiently large  $m$  that is defined solely by the physical properties of the problem, the  $m$ -dimensional volume decays exponentially. After that we prove that the semigroup is uniformly differentiable which, combined with the estimate for the linearized evolution of the infinitesimal volume leads to the conclusion that the Hausdorff dimension of the attractor is finite.

**2. Properties of solutions: Previous results.** In this section we present some pertinent background information from [7] (certain statements are slightly modified and clarified). The following theorem summarizes existence results:

**Theorem 2.1.** *Suppose that the kinetic functions  $g$  satisfies the following assumptions:*

(A1)  $g(u)$  is a continuously differentiable, monotone decreasing, negative function on  $(0, \infty)$  with  $g(0) = -v_0$  for some velocity  $-v_0 < 0$ ;

(A2)  $g(u)$  is sublinear:  $\lim_{u \rightarrow \infty} g(u)/u = 0$ ;

and that the initial data  $u_0(x) \geq 0$ ,  $u_0(x) \in C(-\infty, \infty)$ . Then there exists one and only one classical solution of the free interface problem (1.1)-(1.4). The solution is uniformly bounded and positive for all  $t > 0$ .

The proof is based on the reduction to an integral equation for the interface velocity

$$v(t) = g \left( e^{-\gamma t} \int_{-\infty}^{\infty} G(s(t), t, \xi, 0) u_0(\xi) d\xi - \int_0^t G(s(t), t, s(\tau), \tau) e^{-\gamma(t-\tau)} v(\tau) d\tau \right), \quad (2.1)$$

which arises from the interface condition and the single layer potential representation for the solution operator:

$$\begin{aligned} (Tu_0)(x, t) &:= u(x, t) \\ &= e^{-\gamma t} \int_{-\infty}^{\infty} G(x, t, \xi, 0) u_0(\xi) d\xi - \int_0^t G(x, t, s(\tau), \tau) e^{-\gamma(t-\tau)} v(\tau) d\tau, \end{aligned} \quad (2.2)$$

where

$$G(x, t, \xi, \tau) = \exp\left\{-\frac{(x-\xi)^2}{4(t-\tau)}\right\} [4\pi(t-\tau)]^{-1/2} \quad (2.3)$$

is the heat kernel and  $s(t) = \int_0^t v(\tau) d\tau$ .

**Remark.** The evolution described in the theorem takes place in the cone of positive functions, given  $u_0 \geq 0$ . This fact easily follows either from the integral representation (2.2) or from a comparison principle for parabolic equations.

In the sequel we replace the sublinearity condition (A2) by a stronger condition. We assume that  $g(u)$  is a monotonically decreasing differentiable function on  $[0, \infty]$  with  $|g'| \leq C$  and satisfying

$$-V^0 \leq g(u) \leq -v_0 \text{ for some } V^0, v_0 > 0. \quad (2.4)$$

These conditions are satisfied, for instance, for the standard Arrhenius kinetics where  $v = V^0 \exp(-A/(u - u_\infty))$ .

Under some additional conditions the following smoothness result holds [7]:

**Theorem 2.2.** *Let the initial data  $u_0$  be twice differentiable in  $x < 0$  and  $x > 0$  with bounded derivatives and satisfy the matching condition:*

$$g(u_0(0)) = \frac{\partial u_0^+}{\partial x}(0) - \frac{\partial u_0^-}{\partial x}(0), \quad (2.5)$$

*in addition, let the derivative of the kinetics function  $g'$  be Lipschitz continuous. Then the velocity  $v$  is differentiable.*

**3. A priori estimates in weighted spaces.** For our purposes we need to establish certain a priori bounds on the solution in appropriate weighted spaces that are introduced next. Let  $\omega_\alpha$  be the weight  $\omega_\alpha(x) = e^{\alpha|x|}$ ; we define

$$\|f\|_\alpha = \sup(\omega_\alpha(x)|f(x)|), \quad C_\alpha = \{f \in C(-\infty, \infty) : \|f\|_\alpha < \infty\}$$

We will demonstrate that the global existence results can be extended to  $C_\alpha$  (note the obvious imbedding, if  $\beta > \alpha \geq 0$  then  $C_\beta \subset C_\alpha \subset C(-\infty, \infty)$ ). Similarly we define Hilbert versions of weighted spaces:

$$\|f\|_\alpha = \|\omega_\alpha f\|_{L_2(-\infty, \infty)}, \quad H_\alpha = \{f : \|f\|_\alpha < \infty\}$$

It is easy to see that if  $\beta < \alpha$  then  $C_\alpha \subset H_\beta$ , and for any  $f \in C_\alpha$

$$\|f\|_\beta = \left( \int_{-\infty}^{\infty} \omega_\beta^2 f^2 dx \right)^{1/2} = \left( \int_{-\infty}^{\infty} \frac{\omega_\beta^2}{\omega_\alpha^2} \omega_\alpha^2 f^2 dx \right)^{1/2} \leq \|f\|_\alpha \frac{1}{\sqrt{\alpha - \beta}}. \quad (3.1)$$

To follow the evolution in weighted spaces we recast the problem in the interface-attached coordinate frame. We introduce  $\zeta = x - s(t)$ , then the dependent variable becomes  $\tilde{u}(\zeta, t) = u(\zeta + s(t), t)$ ; with the customary abuse of notation we still use  $u(\zeta, t) = u(\zeta + s(t), t)$ . Then the problem takes the form:

$$\begin{aligned} u_t &= u_{\zeta\zeta} + v(t)u_\zeta - \gamma u, & \zeta \neq 0, \quad t > 0, \\ u(\zeta, 0) &= u_0(\zeta) \geq 0, \\ g[u(0, t)] &= v(t), \\ [u_\zeta(0, t)] &:= u_\zeta^+(0, t) - u_\zeta^-(0, t) = v(t) \end{aligned} \quad (3.2)$$

The representation formula (2.2) can be rewritten in new variables

$$\begin{aligned} (Tu_0)(\zeta, t) &= u(\zeta, t) = e^{-\gamma t} \int_{-\infty}^{\infty} G(\zeta + s(t), t, \xi, 0) u_0(\xi) d\xi \\ &\quad - \int_0^t G(\zeta + s(t), t, s(\tau), \tau) e^{-\gamma(t-\tau)} v(\tau) d\tau, \end{aligned}$$

It is convenient to split the representation formula (2.2) for the semigroup operator  $T$  into two operators: the contribution of the free boundary

$$T_1(t)u_0(\zeta) = - \int_0^t e^{-\gamma(t-\tau)} G(\zeta + s(t), t, s(\tau), \tau) [v(\tau)] d\tau$$

and that of the initial data  $T_2(t)u_0(\zeta) = e^{-\gamma t} \int_{-\infty}^{\infty} G(\zeta + s(t), t, \xi, 0)u_0(\xi)d\xi$  All the norm estimates in the subsequent sections should be understood in the  $\zeta t$ -coordinates. Alternatively the estimates can be carried out in the original coordinates, with the shifted weight function  $\omega(x - s(t))$ . In our estimates in the sequel we consistently follow the latter approach. Thus, everywhere for any function  $F$

$$|F(\cdot, t)|_{\alpha} := \sup_{\zeta} |F(\zeta, t)\omega_{\alpha}(\zeta)| = \sup_{\zeta} |F(\zeta + s(t), t)\omega_{\alpha}(\zeta)| = \sup_x |F(x, t)\omega_{\alpha}(x - s(t))|$$

(Here and below we use a convenient notation " := " which always means "equals by definition").

Below we will frequently encounter integrals of the error-function type. To estimate them we employ the following simple result.

**Lemma 3.1.** *For  $a, b > 0$*

$$\int_a^{\infty} \exp(-b\eta^2)d\eta \leq \begin{cases} \frac{1}{2\sqrt{b}} \exp(-ba^2), & \text{for } a > 1/\sqrt{b} \\ \frac{\sqrt{\pi}}{2\sqrt{b}}, & \text{for } 0 \leq a < 1/\sqrt{b} \end{cases}$$

*Proof.* If  $a\sqrt{b} > 1$  then

$$\int_a^{\infty} \exp(-b\eta^2)d\eta = \frac{1}{\sqrt{b}} \int_{a\sqrt{b}}^{\infty} \exp(-\eta^2)d\eta \leq \int_a^{\infty} \eta \exp(-b\eta^2)d\eta = \frac{1}{2\sqrt{b}} \exp(-ba^2)$$

On the other hand

$$\int_a^{\infty} \exp(-b\eta^2)d\eta \leq \int_0^{\infty} \exp(-b\eta^2)d\eta = \frac{\sqrt{\pi}}{2\sqrt{b}}$$

□

### 3.1. Estimates for the solution: Contribution from initial data.

**Proposition 3.2.** *For sufficiently small  $\alpha$  (if  $\alpha$  satisfies  $\alpha^2 + \alpha V^0 - \gamma < 0$ ) the contribution from the initial data in the  $C_{\alpha}$ -norm decays exponentially in time:*

$$|u_2(\cdot, t)|_{\alpha} \leq 2 \exp[(-\gamma + \alpha^2 + \alpha V^0)t] |u_0|_{\alpha}$$

*Proof.* For the contribution from the initial data,  $u_2(\cdot, t) = T_2(t)u_0$ , we have:

$$\begin{aligned} & \sup_{\zeta} |u_2(\zeta, t)\omega_{\alpha}(\zeta)| = \sup_x \left\{ \omega_{\alpha}(x - s(t)) \left| \int_{-\infty}^{\infty} e^{-\gamma t} G(x, t, \xi, 0)u_0(\xi)d\xi \right| \right\} \\ & = \sup \left\{ e^{-\gamma t} \omega_{\alpha}(x - s(t)) \left| \int_{-\infty}^{\infty} \frac{1}{\omega_{\alpha}(\xi)} G(x, t, \xi, 0)u_0(\xi)\omega_{\alpha}(\xi)d\xi \right| \right\} \\ & \leq \frac{e^{-\gamma t}}{2\sqrt{t\pi}} |u_0|_{\alpha} \sup \{ \omega_{\alpha}(x - s(t)) [ \int_{-\infty}^0 e^{\alpha\xi} \exp(-\frac{(x-\xi)^2}{4t})d\xi \\ & + \int_0^{\infty} e^{-\alpha\xi} \exp(-\frac{(x-\xi)^2}{4t})d\xi ] \} \end{aligned}$$

Each of the integrals should be estimated separately for  $x < 0$  and  $x > 0$  with the maximum of the estimates chosen for the estimate of the norm. At the same time it is easy to see that the two integrals can be transformed into each other through the change of variables  $x \rightarrow -x$ , therefore it suffices to estimate only one of them and double the result.

We proceed as follows

$$\frac{1}{2\sqrt{t}} \int_0^{\infty} e^{-\alpha\xi} \exp(-\frac{(x-\xi)^2}{4t})d\xi = e^{\alpha^2 t} e^{-\alpha x} \int_{-\frac{(x-2t\alpha)}{2\sqrt{t}}}^{\infty} \exp(-\eta^2)d\eta$$

For  $x > 0$  the above expression

$$e^{\alpha^2 t} e^{-\alpha x} \int_{-\frac{(x-2t\alpha)}{2\sqrt{t}}}^{\infty} \exp(-\eta^2) d\eta \leq \sqrt{\pi} e^{\alpha^2 t} e^{-\alpha x}$$

and therefore

$$\begin{aligned} & \frac{e^{-\gamma t}}{\sqrt{\pi}} |u_0|_{\alpha} \sup_{x>0} \omega_{\alpha}(x-s(t)) \sqrt{\pi} e^{\alpha^2 t} e^{-\alpha x} = \\ & |u_0|_{\alpha} \sup_{x>0} \exp(-\gamma t + \alpha^2 t - \alpha x) \exp(\alpha[x-s(t)]) \leq |u_0|_{\alpha} \exp(\alpha^2 - \gamma + \alpha V^0) t \end{aligned}$$

where we have used the bound  $-s(t) \leq V^0 t$ .

To estimate the integral

$$e^{\alpha^2 t} e^{-\alpha x} \int_{\frac{(2t\alpha-x)}{2\sqrt{t}}}^{\infty} \exp(-\eta^2) d\eta$$

for  $x < 0$ , we apply Lemma 3.1. If the lower limit is larger than unity then

$$e^{\alpha^2 t} e^{-\alpha x} \int_{\frac{(2t\alpha-x)}{2\sqrt{t}}}^{\infty} \exp(-\eta^2) d\eta \leq \frac{1}{2} \exp(\alpha^2 t - \alpha x - \frac{(2t\alpha-x)^2}{4t}) = \frac{1}{2} \exp(-\frac{x^2}{4t})$$

If the lower limit is smaller than unity then

$$\begin{aligned} & e^{\alpha^2 t} e^{-\alpha x} \int_{\frac{(2t\alpha-x)}{2\sqrt{t}}}^{\infty} \exp(-\eta^2) d\eta \leq \frac{\sqrt{\pi}}{2} \exp(\alpha^2 t - \alpha x) \\ & = \frac{\sqrt{\pi}}{2} \exp(\frac{(2t\alpha-x)^2}{4t}) \exp(-\frac{x^2}{4t}) \leq \frac{e\sqrt{\pi}}{2} \exp(-\frac{x^2}{4t}) \end{aligned}$$

Thus, for  $x < 0$ , we get

$$\begin{aligned} & \frac{e\sqrt{\pi}}{2} \frac{e^{-\gamma t}}{\sqrt{\pi}} |u_0|_{\alpha} \sup_{x<0} \exp(\alpha|x-s(t)| - \frac{x^2}{4t}) \leq \\ & \frac{e}{2} e^{-\gamma t} |u_0|_{\alpha} \max[\sup_{x<s(t)} \exp\{-\alpha x + \alpha s(t) - \frac{x^2}{4t}\}, \sup_{s(t)<x<0} \exp\{\alpha x - \alpha s(t) - \frac{x^2}{4t}\}] \\ & = \frac{e}{2} e^{-\gamma t} |u_0|_{\alpha} \max[\exp(\alpha^2 t - \alpha v_0 t), \exp(V^0 t \alpha)] \end{aligned}$$

In the above estimate we used the elementary inequality:

$$\begin{aligned} -\alpha x + \alpha s(t) - \frac{x^2}{4t} &= -(\frac{x}{\sqrt{4t}} + \sqrt{\alpha t})^2 + \alpha^2 t + \alpha s(t) \leq \\ & \alpha^2 t + \alpha s(t) \leq \alpha^2 t - \alpha v_0 t \end{aligned}$$

(note that  $s(t) \leq -v_0 t$ ).

Collecting the estimates for all the cases ( $x < 0$ , and  $x > 0$ )

$$\begin{aligned} |u_2(\cdot, t)|_{\alpha} &= |u_0|_{\alpha} e^{-\gamma t} \max[\frac{e}{2} \exp(\alpha^2 t - \alpha v_0 t), \frac{e}{2} \exp(V^0 t \alpha), \exp(\alpha^2 + \alpha V^0) t] \\ &\leq \frac{e}{2} \exp[(-\gamma + \alpha^2 + \alpha V^0) t] |u_0|_{\alpha} < 2 \exp[(-\gamma + \alpha^2 + \alpha V^0) t] |u_0|_{\alpha} \end{aligned}$$

Thus, for any  $\alpha$  we have obtained an a priori estimate on the contribution from the initial data valid for all time. If  $\alpha$  is sufficiently small,  $\alpha^2 + \alpha V^0 - \gamma < 0$ , then the norm of the contribution is exponentially decaying. We also note that for  $\alpha \rightarrow 0$  the estimate has a limit and takes the form

$$|u_2(\cdot, t)|_0 \leq \frac{e}{2} \exp(-\gamma t) |u_0|_0 \quad (3.3)$$

□

### 3.2. Estimates for the solution: Contribution from the free interface.

**Proposition 3.3.** *The  $C_\alpha$ -norm of the contribution from the free interface is uniformly bounded for all time:*

$$|(T_1 u_0)(\cdot, t)|_\alpha \leq V^0 / \sqrt{\gamma},$$

provided  $\alpha < \alpha_{space} := \min(v_0/4, \gamma/(2V^0))$ .

*Proof.* To estimate the free-interface contribution to the solution  $T_1(t)u_0$  behind the interface  $x > s(t)$  we split the interval of integration into two subsets:  $\chi_1 = \{\tau \in [0, t] : s(\tau) < (s(t)+x)/2\}$  and its complement  $\chi_2 = \{\tau \in [0, t] : s(\tau) > (s(t)+x)/2\}$ .

$$|T_1(t)u_0| \leq \int_0^t G(x, t, s(\tau), \tau) e^{-\gamma(t-\tau)} |v(\tau)| d\tau = \int_{\chi_1} + \int_{\chi_2} = I_1 + I_2,$$

For the first integral we have

$$\begin{aligned} I_1 &= \int_{\chi_1} \frac{\exp[-(x-s(\tau))^2 \frac{1}{4(t-\tau)}]}{2\sqrt{\pi(t-\tau)}} e^{-\gamma(t-\tau)} |v(\tau)| d\tau \\ &\leq \frac{V^0}{2\sqrt{\pi}} \int_{\chi_1} (t-\tau)^{-1/2} \exp[-(x-s(t))^2 \frac{1}{16(t-\tau)}] e^{-\gamma(t-\tau)} d\tau \\ &\leq \frac{V^0}{2\sqrt{\pi}} \int_{\chi_1} (t-\tau)^{-1/2} \exp[-(x-s(t)) \frac{v_0}{8}] e^{-\gamma(t-\tau)} d\tau \\ &\leq \frac{V^0}{2\sqrt{\pi}} \exp[-(x-s(t)) \frac{v_0}{4}] \int_0^{(x-s(t))/(2v_0)} \eta^{-1/2} e^{-\gamma\eta} d\eta \\ &= \frac{V^0}{2\sqrt{\gamma}} \operatorname{erf} \left( \sqrt{\gamma} \frac{x-s(t)}{2v_0} \right) \exp[-(x-s(t)) \frac{v_0}{4}] \leq \frac{V^0}{2\sqrt{\gamma}} \exp[-(x-s(t)) \frac{v_0}{4}] \end{aligned}$$

The following inequalities

$$(x-s(\tau))^2 \frac{1}{(t-\tau)} \leq \left(\frac{x-s(t)}{2}\right)^2 \frac{1}{(t-\tau)} \leq \left(\frac{x-s(t)}{2}\right)^2 \frac{2v_0}{x-s(t)}$$

have been used to replace the exponent in the Gaussian kernel, which after that gave the exponential decay factor. We note that the estimate has a regular behavior at the limit  $\gamma \rightarrow 0$  giving the bound

$$\frac{V^0 \sqrt{x-s(t)}}{\sqrt{2v_0\pi}} \exp[-(x-s(t)) \frac{v_0}{4}].$$

It is a manifestation of the fact that the heat loss is immaterial in the vicinity of the interface (cf. the next estimate which shows that the presence of heat losses is essential for decay at large distances from the interface).

For the integral  $I_2$  we use Lemma 3.1 to obtain

$$\begin{aligned}
I_2 &= \int \frac{e^{-\gamma(t-\tau)}}{2\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(x-s(\tau))^2}{4(t-\tau)}\right] |v(\tau)| d\tau \\
&\leq V^0 \int_{(x-s(t))/(2V^0)}^{\infty} \frac{1}{2\sqrt{\pi\eta}} e^{-\gamma\eta} d\eta = \frac{V^0}{\sqrt{\pi}} \int_{\sqrt{(x-s(t))/(2V^0)}}^{\infty} \exp(-\gamma\xi^2) d\xi \\
&\leq \begin{cases} \frac{V^0}{2\sqrt{\gamma\pi}} \exp(-\gamma(x-s(t))/(2V^0)), & \text{for } \gamma(x-s(t))/(2V^0) > 1 \\ \frac{V^0}{2\sqrt{\gamma}}, & \text{for } 0 \leq \gamma(x-s(t))/(2V^0) < 1 \end{cases} \quad (3.4)
\end{aligned}$$

Thus for  $x > s(t)$  we obtain

$$|T_1(t)u_0(x)| \leq \begin{cases} \frac{V^0}{2\sqrt{\gamma}} \exp[-(x-s(t))\frac{v_0}{4}] + \\ \frac{V^0}{2\sqrt{\gamma}} \exp(-\gamma(x-s(t))/(2V^0)), & \text{for } \gamma(x-s(t))/(2V^0) > 1 \\ \frac{V^0}{2\sqrt{\gamma}} \exp[-(x-s(t))\frac{v_0}{4}] + \\ \frac{V^0}{2\sqrt{\gamma}} & \text{for } 0 < \gamma(x-s(t))/(2V^0) < 1 \end{cases} \quad (3.5)$$

We note that upon multiplication by the weight  $\exp(\alpha(x-s(t)))$  the right hand sides of the estimate (3.5) are decaying exponentials, provided

$$\alpha < \alpha_{space} := \min(v_0/4, \gamma/(2V^0)), \quad (3.6)$$

that attain their maximum  $V^0/\sqrt{\gamma}$  at  $x-s(t) = 0$ . Thus

$$|T_1(t)u_0(x)| \leq V^0/\sqrt{\gamma} \text{ for } x > s(t)$$

Ahead of the interface  $x < s(t)$  we have:

$$\begin{aligned}
& \left| \int_0^t e^{-\gamma(t-\tau)} \frac{e^{-(x-s(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)}} v(\tau) d\tau \right| \\
& \leq \int_0^t e^{-\gamma(t-\tau)} \exp\left\{-\frac{[(x-s(t)) + (s(t)-s(\tau))]^2}{4(t-\tau)}\right\} \frac{|v(\tau)| d\tau}{\sqrt{4\pi(t-\tau)}} \\
& \leq \exp\left\{-\frac{v_0|x-s(t)|}{2} - \frac{|x-s(t)|^2}{4t}\right\} \\
& \quad \times \frac{V^0}{\sqrt{\pi}} \int_0^t \exp\left\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)} - \gamma(t-\tau)\right\} \frac{d\tau}{2\sqrt{(t-\tau)}} \\
& \leq \exp\left\{-\frac{v_0|x-s(t)|}{2} - \frac{|x-s(t)|^2}{4t}\right\} \frac{V^0}{\sqrt{\pi}} \int_0^t \exp\left\{\left(-\frac{v_0^2}{4} - \gamma\right)(t-\tau)\right\} \frac{d\tau}{2\sqrt{(t-\tau)}} \\
& \leq \frac{V^0}{\sqrt{v_0^2 + 4\gamma}} \exp\left\{-\frac{v_0|x-s(t)|}{2} - \frac{|x-s(t)|^2}{4t}\right\} \quad (3.7)
\end{aligned}$$

Thus,

$$|T_1(t)u_0(x)| \leq \frac{V^0}{\sqrt{v_0^2 + 4\gamma}} \exp\left\{-\frac{v_0|x-s(t)|}{2}\right\} \text{ for } x < s(t)$$

Now it is easy to obtain the estimate for the norm:

$$|(T_1 u_0)(\cdot, t)|_\alpha \leq \max\left(\sup_{x < s(t)} \left[\frac{V^0}{\sqrt{v_0^2 + 4\gamma}} \exp\left(-\left(\frac{v_0}{2} - \alpha\right)(s(t) - x)\right)\right], \frac{V^0}{\sqrt{\gamma}}\right) \leq \frac{V^0}{\sqrt{\gamma}} \quad (3.8)$$

□

By combining the estimates for the contribution from initial data and that from the free interface we arrive at the following result:

**Theorem 3.4.** For  $0 \leq \alpha \leq \alpha_{space}$

$$|(T u_0)(\cdot, t)|_\alpha \leq \frac{V^0}{\sqrt{\gamma}} + 2 \exp[(-\gamma + \alpha^2 + \alpha V^0)t] |u_0|_\alpha$$

If, in addition,

$$\alpha < \alpha_{time} := \frac{V^0}{2} (\sqrt{1 + 4\gamma/(V^0)^2} - 1) \quad (3.9)$$

(here  $\alpha_{time}$  is the positive root of  $\alpha^2 + \alpha V^0 - \gamma = 0$ ) then the contribution from the initial data decays exponentially.

For the future use we combine the bounds:

$$\alpha_{min} = \min(\alpha_{time}, \alpha_{space}) \quad (3.10)$$

**Remark.** We note that for realistic problems  $\gamma/V^0 \ll 1$  then  $\alpha_{time} \approx \gamma/V^0$ , thus both bounds are of the same order and in this case  $\alpha_{min} = \alpha_{space}$ .

**3.3. Estimates for the derivative: Contribution from initial data.** We also need estimates for the spatial derivative of the solution. We start with the contribution from initial data.

**Proposition 3.5.** For  $\alpha < \alpha_{time}$  the  $C_\alpha$ -norm of the derivative of the contribution from the initial data decays exponentially in time:

$$|(T_2 u)_x(\cdot, t)|_\alpha \leq |u_0|_\alpha \left(\frac{2}{\sqrt{t\pi}} + \frac{\alpha}{2}\right) \exp[(\alpha^2 + \alpha V^0 - \gamma)t].$$

*Proof.* First we split the integral

$$\begin{aligned} |(T_2 u)_x(\cdot, t)|_a &= \sup \left\{ \omega_\alpha(x - s(t)) \left| \int_{-\infty}^{\infty} e^{-\gamma t} G_x(x, t, \xi, 0) u_0(\xi) d\xi \right| \right\} \\ &= \sup \left\{ e^{-\gamma t} \omega_\alpha(x - s(t)) \left| \int_{-\infty}^{\infty} \frac{1}{\omega_\alpha(\xi)} G_x(x, t, \xi, 0) u_0(\xi) \omega_\alpha(\xi) d\xi \right| \right\} \\ &= \frac{e^{-\gamma t}}{2\sqrt{t\pi}} |u_0|_\alpha \sup \left\{ \omega_\alpha(x - s(t)) \left[ \int_0^{\infty} e^{-\alpha\xi} \left| \frac{\partial}{\partial x} \exp\left(-\frac{(x-\xi)^2}{4t}\right) \right| d\xi \right. \right. \\ &\quad \left. \left. + \int_{-\infty}^0 e^{\alpha\xi} \left| \frac{\partial}{\partial x} \exp\left(-\frac{(x-\xi)^2}{4t}\right) \right| d\xi \right] \right\} \end{aligned}$$

Each of the integrals should be estimated separately for  $x < 0$  and  $x > 0$  and the maximum of the estimates should be chosen for the estimate of the norm. At the same time it is easy to see that the two integrals can be transformed to each other through the change  $x \rightarrow -x$ , therefore it suffices to estimate only one of them and double the result.

We proceed as follows. For  $x > 0$  we integrate by parts

$$\begin{aligned}
& \int_0^\infty e^{-\alpha\xi} \left| \frac{\partial}{\partial x} \exp\left(-\frac{(x-\xi)^2}{4t}\right) \right| d\xi = \int_0^\infty e^{-\alpha\xi} \left| \frac{\partial}{\partial \xi} \exp\left(-\frac{(x-\xi)^2}{4t}\right) \right| d\xi \\
& = \int_0^x e^{-\alpha\xi} \frac{\partial}{\partial \xi} \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi - \int_x^\infty e^{-\alpha\xi} \frac{\partial}{\partial \xi} \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi \\
& = \alpha \left( \int_0^x e^{-\alpha\xi} \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi - \int_x^\infty e^{-\alpha\xi} \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi \right) \\
& \quad + 2e^{-\alpha x} - \exp\left(-\frac{x^2}{4t}\right) \\
& = \alpha \left\{ \int_{-x}^0 e^{-\alpha(x+\eta)} \exp\left(-\frac{\eta^2}{4t}\right) d\eta - \int_0^\infty e^{-\alpha(x+\eta)} \exp\left(-\frac{\eta^2}{4t}\right) d\eta \right\} \\
& \quad + 2e^{-\alpha x} - \exp\left(-\frac{x^2}{4t}\right) \\
& \leq \alpha \exp(\alpha^2 t - \alpha x) \int_{-x}^0 \exp\left(-\frac{\eta^2 + 4\alpha\eta t + 4\alpha^2 t^2}{4t}\right) d\eta + 2e^{-\alpha x} \\
& \leq \alpha \exp(\alpha^2 t - \alpha x) \int_{-\infty}^\infty \exp\left(-\frac{(\eta + 2\alpha t)^2}{4t}\right) d\eta + 2e^{-\alpha x} \\
& = \alpha \exp(\alpha^2 t - \alpha x) \sqrt{t\pi} + 2e^{-\alpha x}
\end{aligned}$$

For  $x < 0$  we integrate by parts to obtain

$$\begin{aligned}
& \int_0^\infty e^{-\alpha\xi} \left| \frac{\partial}{\partial x} \exp\left(-\frac{(x-\xi)^2}{4t}\right) \right| d\xi = \int_0^\infty e^{-\alpha\xi} \left| \frac{\partial}{\partial \xi} \exp\left(-\frac{(x-\xi)^2}{4t}\right) \right| d\xi = \\
& - \int_0^\infty e^{-\alpha\xi} \frac{\partial}{\partial \xi} \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi = -\alpha \int_{-x}^\infty e^{-\alpha(x+\xi)} \exp\left(-\frac{\xi^2}{4t}\right) d\xi + \exp\left(-\frac{x^2}{4t}\right) \\
& \leq \exp\left(-\frac{x^2}{4t}\right)
\end{aligned}$$

Now we are ready to estimate the norm:

$$\begin{aligned}
|(T_2 u)_x(\cdot, t)|_\alpha & \leq \frac{e^{-\gamma t}}{2\sqrt{t\pi}} |u_0|_\alpha \\
& \quad \max \left\{ \sup_{x>0} \left\{ \omega_\alpha(x-s(t)) (\alpha\sqrt{t\pi} \exp(\alpha^2 t - \alpha x) + 2e^{-\alpha x}) \right\}, \right. \\
& \quad \left. \sup_{x<0} \left\{ \omega_\alpha(x-s(t)) \exp\left(-\frac{x^2}{4t}\right) \right\} \right\}
\end{aligned}$$

For the term with  $x > 0$  in the above estimate, we have

$$\begin{aligned}
& \frac{e^{-\gamma t}}{2\sqrt{t\pi}} |u_0|_\alpha \sup_{x>0} \left\{ \omega_\alpha(x-s(t)) [\alpha\sqrt{t\pi} \exp(\alpha^2 t - \alpha x) + 2e^{-\alpha x}] \right\} \\
& \leq \frac{e^{-\gamma t}}{2\sqrt{t\pi}} |u_0|_\alpha \exp(-\alpha s(t)) [2 + \alpha\sqrt{t\pi} \exp(\alpha^2 t)] \\
& \leq |u_0|_\alpha \left\{ \frac{1}{\sqrt{t\pi}} \exp[(\alpha V^0 - \gamma)t] + \frac{\alpha}{2} \exp[(\alpha^2 + \alpha V^0 - \gamma)t] \right\}
\end{aligned}$$

For  $x < 0$ , we have

$$\begin{aligned} |(T_2u)_x(\cdot, t)|_\alpha &\leq \frac{e^{-\gamma t}}{2\sqrt{t\pi}} |u_0|_\alpha \sup_{x < 0} \left\{ \omega_\alpha(x - s(t)) \exp\left(-\frac{x^2}{4t}\right) \right\} \\ &\leq \frac{e^{-\gamma t}}{2\sqrt{t\pi}} |u_0|_\alpha \max \left[ \sup_{x < s(t)} \exp\left\{-\alpha x + \alpha s(t) - \frac{x^2}{4t}\right\}, \sup_{s(t) < x < 0} \exp\left\{\alpha x - \alpha s(t) - \frac{x^2}{4t}\right\} \right] \\ &\leq \frac{1}{2\sqrt{t\pi}} |u_0|_\alpha \max \left\{ \exp[(\alpha^2 - \alpha v_0 - \gamma)t], \exp[(\alpha V^0 - \gamma)t] \right\} \end{aligned}$$

In the above estimate we have replaced  $\alpha x - \frac{x^2}{4t}$  by its maximum  $\alpha^2 t$ . We collect the estimate for  $x > 0$  and  $x < 0$  to obtain

$$\begin{aligned} |(T_2u)_x(\cdot, t)|_\alpha &\leq |u_0|_\alpha \left\{ \frac{1}{\sqrt{t\pi}} \exp[(\alpha V^0 - \gamma)t] + \left(\frac{1}{\sqrt{t\pi}} + \frac{\alpha}{2}\right) \exp[(\alpha^2 + \alpha V^0 - \gamma)t] \right\} \\ &\leq |u_0|_\alpha \left(\frac{2}{\sqrt{t\pi}} + \frac{\alpha}{2}\right) \exp[(\alpha^2 + \alpha V^0 - \gamma)t]. \end{aligned} \quad (3.11)$$

□

Thus, for any  $\alpha$  we have obtained an a priori estimate on the derivative of the contribution from initial data, valid for all time. If  $0 \leq \alpha < \alpha_{time}$  then the norm of the contribution is exponentially decaying.

#### 3.4. Estimates for the derivative: Contribution from the interface.

**Proposition 3.6.** *The  $C_\alpha$ -norm of the derivative of the contribution from the free interface is uniformly bounded for all time:*

$$|(T_1u)_x(\cdot, t)|_\alpha \leq \mathcal{M}(v_0, V^0, \alpha, \gamma)$$

provided that  $0 \leq \alpha < \min(\frac{v_0}{8}, \frac{\gamma}{2V^0}) := \alpha'_{\min}$

*Proof. (Ahead of the interface).* The estimate ahead of the front  $x \leq s(t)$  is treated as follows. We consider separately two cases:  $|s(t) - x| > 1$  and  $|s(t) - x| \leq 1$ .

For the case  $|s(t) - x| > 1$

$$\begin{aligned} |(T_1u)_x(x, t)| &= \left| \int_0^t e^{-\gamma(t-\tau)} \frac{x - s(\tau)}{2(t-\tau)} \frac{e^{-(x-s(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)}} v(\tau) d\tau \right| \\ &\leq \left| \int_0^t \frac{(x - s(\tau))^2}{2(t-\tau)(x - s(\tau))} e^{-(x-s(\tau))^2/8(t-\tau)} \times e^{-(x-s(\tau))^2/8(t-\tau)} \frac{v(\tau) d\tau}{\sqrt{4\pi(t-\tau)}} \right| \\ &\leq \left| \int_0^t \frac{4/e}{s(t) - s(\tau)} \exp\left\{-\frac{[(x - s(t)) + (s(t) - s(\tau))]^2}{8(t-\tau)}\right\} \frac{v(\tau) d\tau}{\sqrt{\pi(t-\tau)}} \right| \\ &\leq \frac{4V^0}{v_0 e \sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} \exp\left[-\frac{(x - s(t))^2}{8(t-\tau)} - \frac{v_0}{4} |x - s(t)| - \frac{v_0^2}{8} (t-\tau)\right] d\tau \\ &\leq \frac{4V^0 e^{-v_0|x-s(t)|/4}}{e \sqrt{\pi} v_0 |s(t) - x|} \int_0^t |s(t) - x| (t-\tau)^{-3/2} \exp\left[-\frac{(x - s(t))^2}{8(t-\tau)}\right] d\tau \\ &\leq \frac{\sqrt{256} V^0 e^{-v_0|x-s(t)|/4}}{e \sqrt{\pi} v_0 |s(t) - x|} \int_0^\infty e^{-\eta^2} d\eta \leq \frac{8 V^0 e^{-v_0|x-s(t)|/4}}{e v_0 |s(t) - x|} \end{aligned}$$

In the last estimate we used the following simple observations:  $\xi e^{-\xi} \leq 1/e$ , for  $\xi = \frac{(x - s(\tau))^2}{4(t-\tau)} > 0$ ,  $|s(\tau) - x| > |s(t) - x|$ ,  $|s(\tau) - x| > |s(t) - s(\tau)| > v_0 |t - \tau|$  and substitution  $\eta = |s(t) - x|(t-\tau)^{-1/2}$  to obtain the error function integral.

For the less involved case  $|s(t) - x| \leq 1$  we proceed as follows

$$\begin{aligned}
|(T_1 u)_x(x, t)| &\leq \left| \int_0^t \frac{x - s(\tau)}{2(t - \tau)} \frac{e^{-(x-s(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)}} v(\tau) d\tau \right| \\
&\leq \left| \int_0^t \frac{|x - s(t)| + |s(t) - s(\tau)|}{2(t - \tau)} \frac{e^{-(x-s(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)}} v(\tau) d\tau \right| \\
&\leq \frac{V^0}{\sqrt{\pi}} \int_0^t \frac{|s(t) - x|(t - \tau)^{-3/2}}{4} \exp\left[-\frac{(x - s(t))^2}{4(t - \tau)}\right] d\tau + \\
\frac{(V^0)^2}{4\sqrt{\pi}} \int_0^t \frac{e^{-(s(t)-s(\tau))^2/4(t-\tau)}}{\sqrt{(t-\tau)}} d\tau &\leq \frac{V^0}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2} d\eta + \frac{(V^0)^2}{4\sqrt{\pi}} \int_0^t \frac{e^{-v_0^2(t-\tau)/4}}{\sqrt{(t-\tau)}} d\tau \\
&\leq \frac{V^0}{2} + \frac{(V^0)^2}{2\sqrt{\pi}} \int_0^\infty e^{-v_0^2\eta^2/4} d\eta \leq \frac{V^0}{2} \left(1 + \frac{V^0}{v_0}\right)
\end{aligned}$$

Thus for  $x < s(t)$  we obtain

$$|(T_1 u)_x(x, t)| \leq \begin{cases} \frac{8V^0 \exp(-v_0|x - s(t)|/4)}{ev_0|s(t) - x|} & \text{for } s(t) - x > 1 \\ \frac{V^0}{2} \left(1 + \frac{V^0}{v_0}\right) & \text{for } 0 < s(t) - x < 1 \end{cases} \quad (3.12)$$

□

**Remark.** The proof above shows that if  $|s(t) - s(\tau)| \geq v_0|t - \tau|$  which holds if the basic assumption on the kinetics in (2.4) is satisfied, then the derivative ahead of the interface  $x < s(t)$  decays exponentially

$$|(T_1 u)_x(x, t)| \leq \frac{Ce^{-v_0|x-s(t)|/4}}{|s(t) - x|} \|v\|_{C[0,t]} \quad (3.13)$$

The exponent  $-v_0/4$  can be improved to  $-v_0/(2 + \varepsilon)$  (at the price of increasing  $C$ ).

*Proof. (Behind the interface).* Now we consider the domain behind the interface,  $x > s(t)$ . We split the interval of integration into two subsets:  $\chi_1 = \{\tau \in [0, t] : s(\tau) < (s(t) + x)/2\}$  and its complement  $\chi_2 = \{\tau \in [0, t] : s(\tau) > (s(t) + x)/2\}$ .

$$|(T_1 u)_x(x, t)| \leq \int_0^t \frac{|x - s(\tau)|}{2(t - \tau)} G(x, t, s(\tau), \tau) e^{-\gamma(t-\tau)} |v(\tau)| d\tau = \int_{\chi_1} + \int_{\chi_2} = I_1 + I_2,$$

For the first integral we have

$$\begin{aligned}
I_1 &= \frac{1}{4\sqrt{\pi}} \int_{\chi_1} \frac{|x - s(\tau)|}{(t - \tau)^{3/2}} \exp\left[-\frac{(x - s(\tau))^2}{4(t - \tau)}\right] e^{-\gamma(t-\tau)} |v(\tau)| d\tau \\
&\leq \frac{V^0}{4\sqrt{\pi}} \int_{\chi_1} \frac{|x - s(t)|}{(t - \tau)^{3/2}} \exp\left[-\frac{(x - s(t))^2}{16(t - \tau)}\right] e^{-\gamma(t-\tau)} d\tau \\
&\leq \frac{V^0}{4\sqrt{\pi}} \int_0^{(x-s(t))/(2v_0)} \frac{|x - s(t)|}{(t - \tau)^{3/2}} \exp\left[-\frac{(x - s(t))^2}{16(t - \tau)}\right] e^{-\gamma(t-\tau)} d(t - \tau) \\
&= \frac{V^0}{2\sqrt{\pi}} \int_{\sqrt{(x-s(t))2v_0}}^\infty e^{-\eta^2/16} d\eta
\end{aligned}$$

where  $\eta = (x - s(t))/\sqrt{t - \tau}$ .

To estimate the last integral we apply Lemma 3.1 to obtain

$$I_1 \leq \begin{cases} \frac{2V^0}{\sqrt{\pi}} \exp(-(x - s(t))v_0/8), & \text{for } (x - s(t))2v_0 > 1 \\ 2V^0, & \text{for } 0 \leq (x - s(t))2v_0 < 1 \end{cases}$$

For the estimate on the  $C_\alpha$ -norm, the function will be multiplied by the weight. We note that upon multiplication by  $\exp(\alpha(x - s(t)))$  the right hand sides of the top estimate is a decaying exponential, provided  $\alpha < v_0/8$ , that attain its maximum  $2V^0/\sqrt{\pi}$  at  $x - s(t) = 0$ , while the bottom term is bounded by  $2V^0 \exp(\alpha/(v_0/8)) < 2eV^0$ . Therefore the contribution of  $I_1$  into the norm is bounded above, for example, by  $6V^0$ :  $I_1 < 6V^0$ .

For the integral  $I_2$

$$I_2 = \int_{\chi_2} \frac{|x - s(\tau)|}{2(t - \tau)} \frac{\exp[-\frac{(x - s(\tau))^2}{4(t - \tau)}]}{2\sqrt{\pi(t - \tau)}} e^{-\gamma(t - \tau)} |v(\tau)| d\tau$$

we use simple geometric considerations that show that in the domain  $\chi_2$  if  $s(\tau) > x$  then

$$\frac{|x - s(\tau)|}{2(t - \tau)} \leq \frac{|s(t) - s(\tau)|}{2(t - \tau)} \leq \frac{V^0}{2}$$

while for  $s(\tau) \leq x$

$$\frac{|x - s(\tau)|}{2(t - \tau)} \leq \frac{1}{2} \left| x - \frac{s(t) + x}{2} \right| / \left( \left| s(t) - \frac{s(t) + x}{2} \right| / V^0 \right) = \frac{V^0}{2}.$$

Therefore the estimate for the derivative in this case reduces to the estimate for the function itself (3.4) and yields:

$$I_2 \leq \frac{V^0}{2} \begin{cases} \frac{V^0}{2\sqrt{\gamma\pi}} \exp(-\gamma(x - s(t))/(2V^0)), & \text{for } \gamma(x - s(t))/(2V^0) > 1 \\ \frac{V^0}{2\sqrt{\gamma}}, & \text{for } 0 \leq \gamma(x - s(t))/(2V^0) < 1 \end{cases}$$

Similarly to the argument for  $I_1$ , one can see that if  $\alpha < \gamma/(2V^0)$  then the contribution of  $I_2$  into the  $C_\alpha$ -norm is bounded by  $V^0 e/(2\sqrt{\gamma})$ .

Thus for  $x > s(t)$  we obtain

$$\sup_{x > s(t)} |(T_1 u)_x(x, t) \exp(\alpha(x - s(t)))| \leq V^0 e/(2\sqrt{\gamma}) + 6V^0 < V^0(2/\sqrt{\gamma} + 6) \quad (3.14)$$

if  $\alpha < \min(\frac{v_0}{8}, \frac{\gamma}{2V^0})$ .

Similarly, if  $\alpha < v_0/4$  then by employing (3.12) we see that

$$\sup_{x < s(t)} |(T_1 u)_x(x, t) \exp(-\alpha(x - s(t)))| \leq \max\left[\frac{8V^0}{ev_0}, \frac{V^0}{2}\left(1 + \frac{V^0}{v_0}\right)e^\alpha\right]$$

Finally, for the norm we get

$$|(T_1 u)_x(\cdot, t)|_\alpha \leq V^0 \max\left[\frac{8}{ev_0}, \frac{1}{2}\left(1 + \frac{V^0}{v_0}\right)e^\alpha, 2/\sqrt{\gamma} + 6\right] := \mathcal{M}(v_0, V^0, \alpha, \gamma) \quad (3.15)$$

The estimate holds if

$$0 \leq \alpha < \min\left(\frac{v_0}{8}, \frac{\gamma}{2V^0}\right) := \alpha'_{\min} \quad (3.16)$$

(Recall that  $T_1 u$  is the contribution from the free interface and therefore the absolute bound on its derivative is independent of the initial data).  $\square$

**Remark.** The choice of the factor  $1/2$  for the split of the domain of integration into two parts above is rather arbitrary; by choosing a factor approaching 1 we can improve the exponent of decay in the subsequent result, the corresponding coefficient though will increase. Consequently, the value  $\frac{v_0}{8}$  in the definition of  $\alpha'_{\min}$  can be improved to become  $\frac{v_0}{4 + \varepsilon}$ .

**3.5. A priori estimates in  $C_\alpha^1$  and  $H_\alpha^1$ .** We use Proposition 3.6 and the estimate in (3.11) to obtain the following result:

**Theorem 3.7.**

$$|(Tu_0)_x(\cdot, t)|_\alpha \leq \mathcal{M} + |u_0|_\alpha \left( \frac{2}{\sqrt{t\pi}} + \frac{\alpha}{2} \right) \exp[(\alpha V^0 + \alpha^2 - \gamma)t]$$

**Remark.** It is easy to demonstrate via integration by parts that if the initial data  $u_0 \in C_\alpha^1$  and satisfy the compatibility condition  $[(u_0)_x]_{x=0} = g(u_0(0))$  then the  $1/\sqrt{t}$  singularity in the above estimate will not take place. The estimate in this case reduces to the estimate for the solution through the derivative of the initial conditions.

We note that a weaker result holds for Hilbert norms.

**Theorem 3.8.** *Let  $u_0 \in C_\alpha$  then the solution satisfies  $u(\cdot, t) \in C((0, \infty), H_\beta^1)$  where  $\beta < \alpha$  and*

$$\begin{aligned} \|(Tu_0)(\cdot, t)\|_{\beta,1} &\leq \frac{1}{\sqrt{\alpha - \beta}} \left\{ \mathcal{M} + |u_0|_\alpha \left( \frac{2}{\sqrt{t\pi}} + \frac{\alpha}{2} \right) \exp[(\alpha V^0 + \alpha^2 - \gamma)t] \right. \\ &\quad \left. + \frac{V^0}{\sqrt{\gamma}} + 2|u_0|_\alpha \exp[(-\gamma + \alpha^2 + \alpha V^0)t] \right\} \end{aligned}$$

*Proof.* The proof is very simple since both  $Tu_0(\cdot, t)$  and  $(Tu_0)_x(\cdot, t) \in H_\beta$  by virtue of the imbedding estimate (3.1).  $\square$

**4. Well-posedness.** In this section we prove that solutions of the free boundary problem in the front-attached coordinate system depend continuously on initial data. This result is used in the sequel to demonstrate smoothness of the elements of the attractor.

**Theorem 4.1.** *In  $C_\alpha$ ,  $0 \leq \alpha < \alpha_{space}$ , solutions of the problem (1.1)-(1.4) depend on initial conditions uniformly continuously. More precisely, if  $\{u(x, t), s(t)\}$  and  $\{\tilde{u}(x, t), \tilde{s}(t)\}$ ,  $0 < t < \sigma$ , are solutions with initial data  $u^0, \tilde{u}^0 \in C_\alpha$ , where  $\sigma > 0$  depends only on the norm  $|u^0|_\alpha$  and  $|\tilde{u}^0|_\alpha$ , then for  $0 < t < \sigma$*

$$\sup_{0 < t < \sigma} |V(t) - \tilde{V}(t)| < c|u^0 - \tilde{u}^0|_\alpha, \quad (4.1)$$

$$|u(\cdot - s(t), t) - \tilde{u}(\cdot - \tilde{s}(t), t)|_\alpha < c|u^0 - \tilde{u}^0|_\alpha \quad (4.2)$$

**Remark.** We state and prove continuous dependence on initial conditions only locally in time. The argument extending this result to any fixed time is based on the a priori estimates and follows closely the proof of global existence.

**Remark.** For simplicity of presentation we include the proof only for the uniform norm  $\alpha = 0$ . The modifications for the case  $\alpha > 0$  are rather routine but somewhat lengthy and follow along the similar lines. Everywhere in the proof below we use the notation  $\|\cdot\| = |\cdot|_0$

*Proof.* The proof consists of two parts. First we establish continuity of the interface velocity. We will establish first the estimate in (4.1) and then use it to derive (4.2). Let  $v$  and  $\tilde{v}$  be solutions of the integral equation in (2.1) with initial data  $u^0$  and  $\tilde{u}^0$  respectively.

$$\begin{aligned} & \|v - \tilde{v}\| \\ &= \left\| g \left( e^{-\gamma t} \int_{-\infty}^{\infty} G(s(t), t, \xi, 0) u_0(\xi) d\xi - \int_0^t G(s(t), t, s(\tau), \tau) e^{-\gamma(t-\tau)} v(\tau) d\tau \right) \right. \\ & \quad \left. - g \left( e^{-\gamma t} \int_{-\infty}^{\infty} G(\tilde{s}(t), t, \xi, 0) \tilde{u}_0(\xi) d\xi - \int_0^t G(\tilde{s}(t), t, \tilde{s}(\tau), \tau) e^{-\gamma(t-\tau)} \tilde{v}(\tau) d\tau \right) \right\| \\ & \leq L e^{-\gamma t} \left\| \int_{-\infty}^{\infty} G(s(t), t, \xi, 0) u_0(\xi) d\xi - \int_{-\infty}^{\infty} G(\tilde{s}(t), t, \xi, 0) \tilde{u}_0(\xi) d\xi \right\| \\ & + L \left\| \int_0^t G(s(t), t, s(\tau), \tau) e^{-\gamma(t-\tau)} v(\tau) d\tau - \int_0^t G(\tilde{s}(t), t, \tilde{s}(\tau), \tau) e^{-\gamma(t-\tau)} \tilde{v}(\tau) d\tau \right\| \end{aligned}$$

To continue the estimate we employ a "coordinate descent":

$$\begin{aligned} & \leq L e^{-\gamma t} \left\| \int_{-\infty}^{\infty} (G(s(t), t, \xi, 0) - G(\tilde{s}(t), t, \xi, 0)) u_0(\xi) d\xi \right\| \\ & \quad + L e^{-\gamma t} \left\| \int_{-\infty}^{\infty} G(\tilde{s}(t), t, \xi, 0) (u_0(\xi) - \tilde{u}_0(\xi)) d\xi \right\| \\ & \quad + L \left\| \int_0^t G(s(t), t, s(\tau), \tau) e^{-\gamma(t-\tau)} (v(\tau) - \tilde{v}(\tau)) d\tau \right\| \\ & + L \left\| \int_0^t [G(s(t), t, s(\tau), \tau) - G(\tilde{s}(t), t, \tilde{s}(\tau), \tau)] e^{-\gamma(t-\tau)} \tilde{v}(\tau) d\tau \right\| \\ & \quad := L e^{-\gamma t} D_1 + L e^{-\gamma t} D_2 + L D_3 + L D_4 \end{aligned}$$

To estimate the first summand

$$D_1 = \int_{-\infty}^{\infty} (G(s(t), t, \xi, 0) - G(\tilde{s}(t), t, \xi, 0)) u_0(\xi) d\xi := \int_{-\infty}^{\infty} \delta G u_0 d\xi$$

we note that by the mean value theorem,

$$\delta G = (s - \tilde{s}) \frac{\partial G}{\partial x}(s' - \xi, t, 0, 0) = (s - \tilde{s}) \frac{s' - \xi}{2t} G(s' - \xi, t, 0, 0)$$

where

$$s' = s'(t, \xi), \quad s(t) \leq s' \leq \tilde{s}(t).$$

Thus

$$\begin{aligned} & |s' - \xi| G(s' - \xi, t, 0, 0) = (2\sqrt{\pi})^{-1} |s' - \xi| t^{-1/2} e^{-(s' - \xi)^2 / 4t} \\ &= (2\sqrt{\pi})^{-1} (8t)^{1/2} \frac{|s' - \xi|}{(8t)^{1/2}} e^{-(s' - \xi)^2 / 8t} 2^{1/2} (2t)^{-1/2} e^{-(s' - \xi)^2 / 8t} \\ &\leq 4t^{1/2} c_1 G(s' - \xi, 2t, 0, 0) \leq 4c_1 t^{1/2} G(s - \xi, 2t, 0, 0) \end{aligned}$$

where  $c_1 = (2\sqrt{\pi})^{-1} \max(xe^{-x^2})$ . Therefore

$$\begin{aligned} |D_1| &= \left| \int_{-\infty}^{\infty} \delta G u^0 d\xi \right| \leq \frac{s-\tilde{s}}{2t} 4c_1 t^{1/2} \int_{-\infty}^{\infty} G(s-\xi, 2t, 0, 0) |u^0(\xi)| d\xi \\ &\leq 2c_1 t^{1/2} \frac{1}{t} \int_0^t |v - \tilde{v}| d\tau \int_{-\infty}^{\infty} G(s-\xi, 2t, 0, 0) |u^0(\xi)| d\xi \leq 2c_1 t^{1/2} \sup |u^0(\xi)| \|v - \tilde{v}\|. \end{aligned}$$

Obviously,

$$|D_2| \leq \int_{-\infty}^{\infty} G(\tilde{s}(t), t, \xi, 0) |u_0(\xi) - \tilde{u}_0(\xi)| d\xi \leq \|u_0 - \tilde{u}_0\|$$

For the estimate of  $D_3$  we replace the exponentials by 1 and integrate to obtain:

$$|D_3| \leq \frac{1}{\sqrt{\pi}} t^{1/2} \|v - \tilde{v}\|$$

Finally, for  $D_4$

$$\begin{aligned} &\left| \int_0^t [G(s(t), t, s(\tau), \tau) - G(\tilde{s}(t), t, \tilde{s}(\tau), \tau)] e^{-\gamma(t-\tau)} \tilde{v}(\tau) d\tau \right| \\ &\leq \|\tilde{v}\| \int_0^t |\Delta G| e^{-\gamma(t-\tau)} \tilde{v}(\tau) d\tau \tilde{v}(\tau) \end{aligned}$$

The estimations are quite elementary and are based on the mean value theorem. First we note that

$$\begin{aligned} |\Delta G| &:= |G(s(t), t, s(\tau), \tau) - G(\tilde{s}(t), t, \tilde{s}(\tau), \tau)| \\ &= |G(s(t) - s(\tau), t - \tau, 0, 0) - G(\tilde{s}(t) - \tilde{s}(\tau), t - \tau, 0, 0)| \\ &= |s(t) - s(\tau) - (\tilde{s}(t) - \tilde{s}(\tau))| \left| \frac{\partial G}{\partial x}(s', t - \tau, 0, 0) \right| \\ &= \left| \frac{s(t) - \tilde{s}(t) - (s(\tau) - \tilde{s}(\tau))}{2(t - \tau)} \right| |s' G(s', t - \tau, 0, 0)| \\ &= \frac{1}{2} \left| \frac{ds}{dt}(\tau') - \frac{d\tilde{s}}{dt}(\tau') \right| |s' G(s', t - \tau, 0, 0)| \\ &\leq \frac{1}{2} \|v - \tilde{v}\| |s' G(s', t - \tau, 0, 0)| \end{aligned}$$

where  $\tau \leq \tau' \leq t$  and  $s'$  is between  $\tilde{s}(t) - \tilde{s}(\tau)$  and  $s(t) - s(\tau)$ . Since

$$|s'| \leq \max\{|\tilde{s}(t) - \tilde{s}(\tau)|, |s(t) - s(\tau)|\} \leq V_0(t - \tau).$$

and  $|G| \leq C_0(t - \tau)^{-1/2}$  we get the estimate

$$|\Delta G| \leq C_0 V_0 \|v - \tilde{v}\| (t - \tau)^{1/2}$$

Thus, for  $D_4$  we get

$$|D_4| \leq C_4 \|v - \tilde{v}\| t^{3/2}$$

We collect the estimates for  $D_1$ -  $D_4$  to obtain

$$\|v - \tilde{v}\| \leq \|u_0 - \tilde{u}_0\| + C_1 t^{1/2} \|u^0\| \|v - \tilde{v}\| + C_4 \|v - \tilde{v}\| t^{3/2} + C_2 t^{1/2} \|v - \tilde{v}\|$$

From this inequality we see that for  $t < \sigma$ , where  $\sigma$  is small enough

$$\|v - \tilde{v}\| \leq \frac{\|u_0 - \tilde{u}_0\|}{1 - C_1 t^{1/2} \|u^0\| - C_4 t^{3/2} - C_2 t^{1/2}} \leq 2 \|u_0 - \tilde{u}_0\|$$

We note that the value of  $\sigma$  depends only on  $\|u^0\|$  therefore this result can be interpreted as uniformly continuous dependence of  $v$  on the initial data on any ball  $\|u^0\| \leq R$ .

Thus, the estimate (4.1) has been demonstrated. Next we note that

$$\begin{aligned} & |u(x - s(t), t) - \tilde{u}(x - \tilde{s}(t), t)| \\ & \leq |u(x - \tilde{s}(t), t) - \tilde{u}(x - \tilde{s}(t), t)| + |u(x - s(t), t) - u(x - \tilde{s}(t), t)| \end{aligned} \quad (4.3)$$

The second term in the inequality above can be estimated via the mean value theorem:

$$\begin{aligned} |u(x - s(t), t) - u(x - \tilde{s}(t), t)| &= |u_x(x - s', t)[s(t) - \tilde{s}(t)]| \\ &\leq |u_x(x - s', t)| t \|v - \tilde{v}\| \end{aligned}$$

where  $s'$  is the intermediate value between  $s(t)$  and  $\tilde{s}(t)$ .

The estimates for the first term in (4.3) are based on the maximum principle. After the free interfaces are determined, both  $u$  and  $\tilde{u}$  solve the heat equation off their respective boundaries. Their difference  $w = u - \tilde{u}$  satisfies the heat equation in each of the three domains

$$x < s(\tau), \quad s(\tau) < x < \tilde{s}(\tau), \quad \tilde{s}(\tau) < x, \quad \tau < \sigma$$

here we assumed that  $s(\tau) < \tilde{s}(\tau)$  and that  $\sigma$  is such that the inequality holds for all  $\tau < \sigma$ . It is easy to estimate the boundary values of the difference

$$\begin{aligned} |w((s(t), t))| &= |u(s(t), t) - \tilde{u}(s(t), t)| \\ &= |g^{-1}(v(t)) - g^{-1}(\tilde{v}(t)) - \tilde{u}_x(s', t)[s(t) - \tilde{s}(t)]| \\ &\leq L|v(t) - \tilde{v}(t)| + |\tilde{u}_x(s', t)| t \|v - \tilde{v}\| \end{aligned}$$

where again  $s'$  is the intermediate value between  $s(t)$  and  $\tilde{s}(t)$ . Because of the a priori estimate on the derivative

$$|\tilde{u}_x(s', t)| \leq \mathcal{M} + |\tilde{u}_0| \frac{c}{\sqrt{t}}$$

(see Theorem 3.7) we obtain

$$|w((s(t), t))| \leq \|v - \tilde{v}\| (L + \mathcal{M}t + c\sqrt{t} \|\tilde{u}_0\|).$$

A similar estimate holds for the other interface. Of course the initial data for  $w$  is equal to  $u_0 - \tilde{u}_0$ . From the maximum principle for each of the three domains we obtain that

$$|u(., t) - \tilde{u}(., t)| \leq C_1 \|v - \tilde{v}\| + C_2 \|u_0 - \tilde{u}_0\|$$

□

**5. Absorbing set and attractor.** In this section we use the estimates obtained above to establish existence of bounded absorbing sets and of the attractor which is compact in the weighted space of continuous functions. In order to establish compactness of the attractor we need to make use of the heat losses. It can be verified that most of the estimates and analytical properties of the solutions can be obtained without the heat losses. The presence of heat losses only improves the estimates. On the other hand the problem with the heat losses exhibits uniform exponential decay in time of the contribution of initial data which is utilized in the proof of compactness of the attractor. We remind that the semigroups discussed below are defined for the problem in the interface-attached coordinate system.

For our purposes we rephrase Theorem 3.4 in the following form:

**Proposition 5.1.** *Let  $0 \leq \alpha < \alpha_{time}$  (where  $\alpha_{time}$  is defined in (3.9)), then:*

(i) *The semigroup  $T_2$  is uniformly exponentially contracting in  $C_\alpha$ :*

$$r_X(t) = \sup_{u^0 \in X} |T_2(t)u^0|_\alpha \leq C \exp(-\kappa(\gamma, \alpha)t)N, \quad \kappa > 0$$

where  $\kappa(\gamma, \alpha) = \gamma - \alpha^2 - \alpha V^0 > 0$  for any ball

$$X = \{u \in C_\alpha; \quad |u|_\alpha \leq N\}$$

(ii) *For any  $\varepsilon > 0$ , the ball  $B_a := \{u \in C_\alpha : |u|_\alpha \leq \frac{V^0}{\sqrt{\gamma}} + \varepsilon\}$  is an absorbing set for bounded subsets of  $C_\alpha$ . Here the radius  $a$  of the absorbing ball reflects the contribution of the free interface alone.*

Next we prove that the boundary contribution to the evolution, i.e. the operators  $T_1(t)$  are *uniformly compact*. Namely, the following proposition holds:

**Proposition 5.2.** *If  $\alpha < \alpha_{space} = \min(\frac{v_0}{4}, \frac{\gamma}{2V^0})$  then for any  $t_0 > 0$  the orbit of the ball  $\cup_{t \geq t_0} T_1(t)X$  is relatively compact in  $C_\alpha$ .*

*Proof.* For the version of Arzela-Ascoli theorem appropriate for  $C_\alpha$  it is sufficient to have uniform boundedness for the derivative and uniform decay of the family of functions as  $|x'| \rightarrow \infty$  which is faster than the decay prescribed by the weight. From the estimate (3.8) we see that the contributions from the interface decay as  $\exp(-(\alpha_{space} + \varepsilon)|x|)$  which can be made faster than any  $\exp(-\alpha|x|)$  for  $\alpha < \alpha_{space}$ . On the other hand, the weighted estimate (3.15) (cf. also Remark after Proposition 3.6) demonstrate that the spatial derivative is uniformly bounded. Then it is easy to construct a finite  $\varepsilon$ -net by choosing a finite interval beyond which the functions of the family are smaller than  $\varepsilon$  and extending the elements of the  $\varepsilon$ -net from this interval by zero.  $\square$

The properties of the evolution operator  $T(t)$  described in the above propositions allow us to apply the abstract general result (see, for example, [24, Chap. 1]) that in our situation can be stated as follows:

**Theorem 5.3.** *The  $\omega$ -limit set  $\mathcal{A}_\alpha$  of the absorbing set  $B_a$  is a global compact attractor for the metric space  $C_\alpha$ ;  $\mathcal{A}_\alpha$  is the maximal attractor in  $C_\alpha$  and it is connected.*

In order to demonstrate extra regularity of the elements of the attractor, we shall need a general fact concerning compact attractors (cf. [5]).

**Theorem 5.4.** *Let  $X$  be a Banach space and  $B \subset X$  be a ball. Let  $S : B \rightarrow B$  be a uniformly continuous mapping and  $\mathcal{C} \subset B$  a compact attractor for the iteration semigroup  $S^n$ . Then on the attractor  $S$  is a mapping onto  $SC = \mathcal{C}$ .*

The theorem holds for either continuous or discrete time. For the simplicity of presentation we consider only the discrete case here.

*Proof.* Suppose  $SC \neq \mathcal{C}$ , then there exists  $x_0 \in \mathcal{C}$  and  $x_0 \notin SC$ . In this case there exists a whole ball  $B_r(x_0) = \{x \in \mathcal{C} : \|x - x_0\| < r\}$  such that  $B_r(x_0) \cap SC = \emptyset$ . Indeed, since the attractor is compact, its continuous image is compact and therefore the distance to  $x_0$ , being a continuous function on a compact, attains its nonzero minimum  $r$  on  $SC$ .

We know that  $S$  is uniformly continuous, therefore for any  $\varepsilon$  there exists  $\delta(\varepsilon)$  so that  $\|Sx - Sy\| < \varepsilon$  if  $\|x - y\| < \delta$ .

Since  $x_0 \in \mathcal{C}$  for any  $\varepsilon$  there exists  $n$  so that  $\|S^n x - x_0\| < \varepsilon$  for some  $x \in B$ , simultaneously we can always select  $n$  so large that  $\text{dist}(S^{n-1}B, \mathcal{C}) < \delta$ . Now take  $y = S^{n-1}x$  and  $a \in \mathcal{C}$ ,  $\|y - a\| < \delta$ , then  $\|Sa - x_0\| \leq \|Sa - Sy\| + \|Sy - x_0\| \leq \varepsilon + \varepsilon$ . By selecting  $\varepsilon < r/2$  we come to a contradiction.  $\square$

As an immediate application of the above theorem we obtain the following regularity result:

**Theorem 5.5.** *For  $0 \leq \alpha < \alpha'_{\min}$  (where  $\alpha'_{\min}$  is defined in (3.16)) the semigroup on the attractor  $\mathcal{A}_\alpha$  in the space  $C_\alpha$  is onto;  $\mathcal{A}_\alpha$  consists of differentiable functions that satisfy the estimates*

$$|\phi|_\alpha \leq \frac{V^0}{\sqrt{\gamma}}, \quad |\phi_x|_\alpha \leq \mathcal{M}(v_0, V^0, \alpha, \gamma) \leq \mathcal{M}(v_0, V^0, \alpha'_{\min}, \gamma)$$

which yields that all  $\mathcal{A}_\alpha \subseteq C_\beta^1$  for  $0 \leq \alpha \leq \beta = \alpha'_{\min} - \varepsilon$ .

*Proof.* The theorem follows from the previous one if we note that the required uniform continuity follows from the well-posedness Theorem 4.1. Since the mapping is onto, given  $\phi \in \mathcal{A}_\alpha$  for any  $t$  there exist  $\psi \in \mathcal{A}_\alpha$ , so that  $\phi = T(t)\psi$ . By using estimates (3.8)-(3.15) and taking into account exponential decay of the contribution from initial data as  $t \rightarrow \infty$  (3.3)-(3.11) we obtain the result.  $\square$

**Remark.** Since any function in the attractor can be viewed as a result of evolution by the semigroup, it therefore locally satisfies the heat equation and consequently it is locally  $C^\infty$ . In addition we can show that due to the differentiability of the velocity of the interface, functions in the attractor are  $C_\alpha^3$  up to the interface.

In addition to being onto, the semigroup  $T(t)$  is also one-to-one on the attractor:

**Proposition 5.6.** *The semigroup  $T(t)$  is one-to-one on the attractor  $\mathcal{A}_\alpha$ .*

*Proof.* Let  $T(t)u_1 = T(t)u_2$ . Denote  $t_0 = \inf\{t : (T(t)u_1)(x) \equiv (T(t)u_2)(x)\}$ ; obviously  $t_0 > 0$ . Then the difference  $w = T(t)u_1 - T(t)u_2$  is identically 0 for  $t \geq t_0$ . Let  $t_1 < t_0$  then there exists  $x_0$  such that  $w(x_0) \neq 0$ . next we select the parabolic neighborhood (a cup)  $U(x_0) = \{(x, t) : t_1 \leq t < t_0 + \varepsilon, x_0 - \delta - k(t - t_1) < x < x_0 - \delta + k(t - t_1)\}$  where  $\varepsilon$ , and  $\delta$  and  $k$  are selected in such a way that the neighborhood does not intersect the free interface. Since  $w$  is a solution of the heat equation in  $U(x_0)$  it is real analytic in  $t$  and therefore should be identically 0 in  $U(x_0)$  in contradiction with the assumption  $w(x_0) \neq 0$ .  $\square$

Computations for the volume evolution and the Hausdorff dimension below are implemented in a Hilbert space. It is easy to see that the exponential spatial decay implies the inclusion  $C_\alpha \subset H_0$  see (3.1). Consequently, the compactness result holds for  $H_0$  as well:

**Theorem 5.7.** *Let  $0 < \alpha < \alpha_{\min}$ . Then:*

*The semigroup  $T_2$  is uniformly exponentially contracting in the  $H_0$ -norm:*

$$r_X(t) = \sup_{u^0 \in X} \|T_2(t)u^0\|_{H_0} \leq C \exp(-\kappa(\gamma, \alpha)t)N/\sqrt{\alpha}, \quad \kappa > 0$$

where  $\kappa(\gamma, \alpha) = \gamma - \alpha^2 - \alpha V^0 > 0$  for any ball

$$X = \{u \in C_\alpha; \quad |u|_\alpha \leq N\}.$$

For any  $t_0 > 0$  the orbit of the ball  $\cup_{t \geq t_0} T_1(t)X$  is relatively compact in  $H_0$ .

The  $\omega$ -limit set  $\mathcal{A}_\alpha$  of the absorbing set  $B_\alpha := \{u \in C_\alpha : |u|_\alpha \leq \frac{V^0}{\sqrt{\gamma}} + \varepsilon\}$  is a relatively compact set in the  $H_0$ -metric.

*Proof.* The only additional ingredient of the proof, as compared to the  $C_\alpha$  case is provided by the imbedding estimate (3.1), which yields that the  $\varepsilon$ -net generated for a set in the  $C_\alpha$ -norm is automatically an  $(\varepsilon/\sqrt{\alpha})$ -net in the  $H_0$ -norm. Since the imbedding estimate implies continuity of the imbedding, the above proof essentially repeats the proof of the fact that a continuous image of a compact set is compact.  $\square$

**6. Evolution of the volume elements on the attractor.** In this section we present the main result of the paper which is a proof that the Hausdorff dimension of the attractor is finite (for definiteness we consider  $\mathcal{A}_0$ , i.e.  $\alpha = 0$ ). The proof is based on a study of evolution of the infinitesimal volume along the trajectories in the attractor. We demonstrate that for sufficiently large  $m$  that is defined solely by the physical parameters of the problem the  $m$ -dimensional volume decays exponentially. This property combined with the compactness suggests that the Hausdorff dimension of the attractor for the solutions of the free boundary problem is no larger than  $m$ . In the arguments regarding the Hausdorff dimension of the attractor we follow quite closely the ideas outlined in [24].

We treat the problem in the interface-attached coordinate frame,  $\tilde{x} = x - s(t)$  as a nonlinear evolution problem,

$$\begin{aligned} u_t &= u_{xx} + v(t)u_x - \gamma u := F(u), & -\infty < x < \infty, & \quad x \neq 0 \\ g(u|_{x=0}) &= v(t), & [\partial u / \partial x]|_{x=0} &= v(t), \\ u(x, 0) &= u^0(x). \end{aligned} \tag{6.1}$$

(Tildes have been omitted.)

Let  $\{U(\cdot, t), V(t)\}$  be an orbit in the attractor. Let us consider the formal linearization of the problem (6.1) about  $\{U, V\}$ :

$$z_t = z_{xx} + z_x V - \gamma w - z(0, t)U_x / \nu(V(t)) := F'(U, V)z \tag{6.2}$$

$$z(0, t) + \nu(V(t))[z_x(0, t)] = 0, \tag{6.3}$$

$$z(x, 0) = z_0(x) \tag{6.4}$$

where  $\nu(V) = -(g^{-1})'(V)$ . We require  $\nu(V)$  to be positive and bounded from below;  $\nu(V) \geq \nu_0$ . This condition again mimics the behavior of the Arrhenius kinetics. We have eliminated the velocity perturbation  $v(t)$  in the term  $v(t)U_x$  of the linearization through replacing it by the perturbation of the temperature  $z(0, t)/\nu(V(t))$  that arises from the linearization of the kinetic boundary condition in (6.1). The linearized problem represents the first variation of problem (6.1).

It is possible to show that the linearized problem is well-posed in the following sense:

**Theorem 6.1.** *For any  $z_0 \in H$  there exists a unique solution  $z$  of (6.2-6.4) such that  $z \in L^2(0, T; \Xi(t)) \cap C([0, T]; H)$  where  $\Xi(t) = \{f \in H^1, f(0) + [f_x(0)]\nu(V(t)) = 0\}$*

*Proof.* This linear problem is somewhat nonstandard as it contains a nonlocal term (projection)  $z(0, t)$ . Nonetheless it can be handled as follows. Consider first the problem (6.2-6.4) with a source, and zero initial conditions

$$\begin{aligned}\tilde{w}_t &= \tilde{w}_{xx} + \tilde{w}_x V - \gamma \tilde{w} + \mathcal{F}(x, t) \\ \tilde{w}(0, t) + [\tilde{w}_x(0, t)]\nu(V(t)) &= 0, \quad \tilde{w}(x, 0) = 0,\end{aligned}$$

and let  $\mathfrak{L}$  be its solution operator:  $\tilde{w} = \mathfrak{L}\mathcal{F}(x, t)$ . Existence of unique global solutions for such problems is guaranteed by the general theory of linear parabolic equations.

We regard a solution of (6.2-6.4) as a superposition of an appropriate  $\tilde{w}$  and of  $W(x, t)$  which solves the *homogeneous* problem with the initial condition  $z_0(x)$ . Then, with the nonlocal term viewed as a source, on the boundary one obtains an equation for  $z(0, t)$ :

$$\mathfrak{L}[-z(0, t)U_x/\nu(V(t)) + W(x, t)]_{x=0} = z(0, t) + W(0, t). \quad (6.5)$$

It is not difficult to show that the above equation is uniquely solvable as an integral equation with a sufficiently regular kernel. Thus, the source term is found and, consequently the problem (6.2-6.4) can be solved.  $\square$

In order to estimate the evolution of the volume element we need an estimate for  $\|U_x\|_{H_0}$ . From now on, for brevity the  $H_0$ -norm will be denoted by  $\|\cdot\|$ . From the imbedding estimate (3.1) and estimate (3.15) we obtain the following important result:

**Lemma 6.2.** *For  $U \in \mathcal{A}_0$ ,  $\|U_x\| \leq \mathcal{M}/\sqrt{\alpha'_{\min}} := \mathcal{N}$ , where  $\alpha'_{\min} = \min(\frac{v_0}{8}, \frac{\gamma}{2V^0})$*

We are now ready to estimate the evolution of the volume element. To this end we need to estimate the trace of the finite-dimensional projections of the generator of the linear semigroup. Let  $\{\xi_1, \dots, \xi_m\}$  be  $m$  elements of  $H$  and let  $\{z_1, \dots, z_m\}$  be the corresponding solutions of the linearized problem. Then it can be shown that the volume element spanned by  $\{\xi_1, \dots, \xi_m\}$  evolves accordingly to the formula

$$|z_1(t) \wedge \dots \wedge z_m(t)| = |\xi_1 \wedge \dots \wedge \xi_m| \exp \int_0^t \text{Tr} [F'(U(\tau), V(\tau)) \circ Q_m(\tau)] d\tau,$$

where  $Q_m(\tau) = Q_m(\tau, U, V; \xi_1, \dots, \xi_m)$  is the projector in  $H$  onto the space spanned by  $\Xi(\tau) = \{z_1(\tau), \dots, z_m(\tau)\}$ . In order to calculate the trace we need to choose a basis in  $\Xi(\tau)$  orthogonal in the sense of  $H$ .

Evaluation of the inner product in  $H$  gives rise to sums of integrals over the domain  $(-\infty, 0) \cup (0, \infty)$ . For brevity everywhere in the sequel we denote them by

$$\int_{R^\pm} f(x) dx = \left( \int_{-\infty}^0 + \int_0^\infty \right) f(x) dx. \quad (6.6)$$

Let  $\phi$  be an element of  $\Xi(\tau)$ . Consider the following inner product in  $H$

$$\begin{aligned}\langle F'\phi, \phi \rangle &= -\gamma \langle \phi, \phi \rangle + \int_{R^\pm} \phi_{xx} \phi dx + V \int_{R^\pm} \phi_x \phi dx + [\phi'(0)] \int_{R^\pm} U_x \phi dx \\ &= -\gamma + I_1 + I_2 + I_3\end{aligned} \quad (6.7)$$

We integrate  $I_1$  by parts,

$$I_1 = \phi_x \phi|_{-\infty}^0 + \phi_x \phi|_0^\infty - \int_{R^\pm} \phi_x^2 dx = -[\phi'(0)]\phi(0) - \int_{R^\pm} \phi_x^2 dx$$

It is easily seen that

$$\int_{R^\pm} \phi_x \phi dx = \frac{1}{2} \int_{R^\pm} (\phi^2)' dx = 0$$

then

$$\langle F' \phi, \phi \rangle = -\gamma - [\phi'(0)]\phi(0) - \int_{R^\pm} \phi_x^2 dx + [\phi'(0)] \int_{R^\pm} U_x \phi dx \quad (6.8)$$

In the choice of the  $m$ -dimensional orthonormal set of functions we shall distinguish the two possibilities for the basis element  $\phi$ :  $\phi(0) = 0$  (which defines an  $(m-1)$ -dimensional subspace), and otherwise. Since the trace of the operator is independent of the choice of an orthonormal basis, we can choose  $m-1$  basis elements satisfying the above condition. In the case  $\phi(0) = 0$  we obtain

$$\langle F' \phi, \phi \rangle = -\gamma - \int_{R^\pm} \phi_x^2 dx \leq -\gamma \quad (6.9)$$

Note that the terms with  $[\phi'(0)]$  vanish since  $[\phi'(0)] = -\phi(0)/\nu = 0$  in view of the boundary condition.

For the basis element with  $\phi(0) \neq 0$  the corresponding trace component,

$$\langle F' \phi, \phi \rangle = -\gamma - [\phi_x] \phi|_0 - \int_{R^\pm} \phi_x^2 dx - \frac{\phi(0)}{\nu} \int_{R^\pm} U_x \phi dx$$

is estimated from above as follows. First we estimate the last term:

$$\left| \frac{\phi(0)}{\nu} \int_{R^\pm} U_x \phi dx \right| \leq \frac{1}{2a} \int_{R^\pm} U_x^2 dx + \frac{a}{2} \frac{\phi^2(0)}{\nu^2} \int_{R^\pm} \phi^2 dx = \frac{1}{2a} \int_{R^\pm} U_x^2 dx + \frac{a}{2} \frac{\phi^2(0)}{\nu^2}$$

where  $a > 0$  will be chosen later on.

Now we need the following interpolation result. By integrating from  $-\infty$  to 0 we obtain:

$$\phi^2(0) = \int_{-\infty}^0 (\phi^2)_x dx = 2 \int_{-\infty}^0 \phi \phi_x dx \leq 2 \int_{-\infty}^0 \left( \frac{c\phi_x^2}{2} + \frac{\phi^2}{2c} \right) dx$$

On the other hand,

$$\phi^2(0) = - \int_0^\infty (\phi^2)_x dx = -2 \int_0^\infty \phi \phi_x dx \leq 2 \int_0^\infty \left( \frac{c\phi_x^2}{2} + \frac{\phi^2}{2c} \right) dx.$$

Thus

$$\phi^2(0) \leq \int_{R^\pm} \left( \frac{c\phi_x^2}{2} + \frac{\phi^2}{2c} \right) dx = \int_{R^\pm} \left( \frac{c\phi_x^2}{2} \right) dx + \frac{1}{2c}$$

In some sense, this estimate is a Sobolev trace theorem.

Finally we obtain

$$\begin{aligned} \langle F' \phi, \phi \rangle &\leq -\gamma + \frac{\phi^2(0)}{\nu} - \int_{R^\pm} \phi_x^2 dx + \frac{1}{2a} \int_{R^\pm} U_x^2 dx + \frac{a}{2} \frac{\phi^2(0)}{\nu^2} \\ &\leq -\gamma + \left( \frac{1}{\nu} + \frac{a}{2\nu^2} \right) \frac{1}{2} \left[ c \int_{R^\pm} \phi_x^2 dx + \frac{1}{c} \right] - \int_{R^\pm} \phi_x^2 dx + \frac{1}{2a} \int_{R^\pm} U_x^2 dx \\ &= -\gamma + \left[ \left( \frac{1}{\nu_0} + \frac{a}{2\nu_0^2} \right) \frac{1}{2} c - 1 \right] \int_{R^\pm} \phi_x^2 dx + \frac{1}{2a} \int_{R^\pm} U_x^2 dx + \left( \frac{1}{\nu_0} + \frac{a}{2\nu_0^2} \right) \frac{1}{2c} \end{aligned}$$

If  $a$  and  $c$  are chosen so that the coefficient at the integral of  $\phi_x^2$  is nonpositive, say if  $c = 4\nu_0^2/(2\nu_0 + a)$  then

$$\langle F' \phi, \phi \rangle \leq -\gamma + \frac{1}{2a} \int_{R^\pm} U_x^2 dx + \left( \frac{2\nu_0 + a}{4\nu_0^2} \right)^2$$

for any  $a > 0$ .

By Lemma 6.2 the norm  $\|U_x\| \leq \mathcal{N}$  where the bound depends only on the kinetics. To optimize the estimate above we choose  $a$  that gives the minimum to the expression  $(\frac{2\nu+a}{4\nu^2})^2 + \frac{1}{2a}\mathcal{N}^2$  considered as a function of  $a$ . This results in the estimate

$$\langle F' \phi, \phi \rangle \leq \mu = -\gamma + \min_{a>0} \left( \left( \frac{2\nu_0 + a}{4\nu_0^2} \right)^2 + \frac{1}{2a} \mathcal{N}^2 \right) \leq \left( \frac{2\nu_0 + 1}{4\nu_0^2} \right)^2 + \frac{1}{2} \mathcal{N}^2 \quad (6.10)$$

The explicit form of  $\mu$  is not important for our purposes. Thus, employing the above estimates for the trace entries (6.9), (6.10) we can complete the estimate for the evolution of the volume element:

$$\text{Tr} [F'(U(\tau), V(\tau)) \circ Q_m(\tau)] = \sum_{i=1}^m \langle F' \phi_i, \phi_i \rangle \leq \mu - m\gamma \quad (6.11)$$

Taking  $m > M = \mu/\gamma$  is sufficient for the trace to become negative. Note that  $M$  depends on  $\nu_0$ ,  $\gamma$ ,  $V_0$  and  $v_0$ .

**7. Differentiability of the semigroup.** To utilize the trace estimate developed in the previous section we need to demonstrate that the *nonlinear* evolution of the volume is well approximated by its *linear* counterpart. This will be ensured by the differentiability of the semigroup solving the free-interface problem with respect to the initial conditions, see [24].

For the purposes of this section we need to impose an additional condition on the kinetics function: we will require both  $g$  and  $g^{-1}$  to be twice differentiable. In applications this condition is definitely satisfied for all realistic kinetics.

In Sec. 2 we cited the global existence result for the *classical* solutions of the free interface problem (1.1)-(1.4). However, our trace estimates take place in the geometry of a Hilbert space. Therefore we need to introduce weak solutions by extending the existence theory to more general initial data that belong to a Hilbert space. The scheme of introduction of weak solutions is based on the following estimates:

**Proposition 7.1.** *Let  $U$  and  $W$  be two orbits (i.e., two solutions of the problem (6.1)) with initial data  $U_0, W_0$  in the attractor:  $U = T(t)U_0, W = T(t)W_0$ . Then for any  $t > 0$*

$$\begin{aligned} \|U(t) - W(t)\| &\leq e^{Ct} \|U_0 - W_0\| \\ \int_0^t \|U(t) - W(t)\|_1^2 d\tau &\leq e^{Ct} \|U_0 - W_0\|^2 \end{aligned} \quad (7.1)$$

where  $C$  is a uniform constant.

**Remark.** The above proposition allows us to obtain weak solutions with initial data in the closure of  $\mathcal{A}$  in the  $H$ -norm. Namely, in a standard fashion we select a Cauchy sequence of initial conditions in  $\mathcal{A}$  and define the solution as the corresponding limit of smooth solutions. We note that it is not necessary to take initial data from  $\mathcal{A}$ ; similarly it is possible to define a weak solution for the initial data in the closure of a ball in  $C_\alpha$ .

*Proof.* Let  $U(x, t)$  and  $W(x, t)$  be two solutions of the free boundary problem (in the frame attached to the free boundary)

$$\begin{aligned} U_t &= U_{xx} + [U_x(0, t)]U_x - \gamma U, & g(U(0, t)) &= [U_x(0, t)], & U(x, 0) &= U_0(x), \\ W_t &= W_{xx} + [W_x(0, t)]W_x - \gamma W, & g(W(0, t)) &= [W_x(0, t)], & W(x, 0) &= W_0(x). \end{aligned}$$

The difference  $w = U - W$  solves the following problem

$$\begin{aligned} w_t &= w_{xx} + [U_x(0, t)]w_x + [w_x(0, t)]W_x - \gamma w, \\ -[w_x(0, t)] &= g(W(0, t)) - g(U(0, t)) = -(g(\theta))'w(0, t), \\ w(x, 0) &= U_0(x) - W_0(x). \end{aligned}$$

We also observe that  $-(g(\theta))' \leq \text{const} = C$  while  $U$ ,  $W$  and their  $x$ -derivatives are uniformly bounded on the attractor.

We multiply the equation throughout by  $w$  and integrate to obtain the following energy estimate for the  $H$  norm:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w\|^2 \\ &= \int_{R^\pm} w_{xx} w dx + [U_x(0, t)] \int_{R^\pm} w_x w dx + [w_x(0, t)] \int_{R^\pm} W_x w dx - \gamma \int_{R^\pm} w^2 dx \\ &= -[w_x]w|_0 - \gamma \|w\|^2 - \|w_x\|^2 + [U_x(0, t)] \int_{R^\pm} w_x w dx + [w_x(0, t)] \int_{R^\pm} W_x w dx \end{aligned} \quad (7.2)$$

We need to estimate different terms in (7.2). For the first term we get on respective intervals

$$\begin{aligned} |[w_x(0, t)]w(0, t)| &\leq Cw(0, t)^2 = C \int_{-\infty}^0 (w^2)_x dx \leq 2C \int_{-\infty}^0 |w_x w| dx \\ |[w_x(0, t)]w(0, t)| &\leq Cw(0, t)^2 = -C \int_0^\infty (w^2)_x dx \leq 2C \int_0^\infty |w_x w| dx \end{aligned}$$

The sum of the above inequalities yields the estimate

$$|[w_x(0, t)]w(0, t)| \leq Cw(0, t)^2 \leq C(\varepsilon_1 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2) \quad (7.3)$$

Next,

$$|[U_x(0, t)] \int_{R^\pm} w_x w dx| \leq |[U_x(0, t)]| (\varepsilon_2 \|w\|^2 + \frac{\|w_x\|^2}{\varepsilon_2}) \leq C_1(\varepsilon_2 \|w\|^2 + \frac{1}{\varepsilon_2} \|w_x\|^2)$$

Also,

$$\begin{aligned} & \left| [w_x(0, t)] \int_{R^\pm} W_x w dx \right| \leq C_3 |w(0, t)| \|W_x\| \|w\| \\ & \leq C_3 |w(0, t)| \mathcal{N} \|w\| \leq C_4 (\varepsilon_3 \|w\|^2 + \frac{1}{\varepsilon_3} \|w_x\|^2)^{1/2} \|w\| \\ & \leq C_4 (\sqrt{\varepsilon_3} \|w\| + \frac{1}{\sqrt{\varepsilon_3}} \|w_x\|) \|w\| = C_4 (\sqrt{\varepsilon_3} \|w\|^2 + \frac{1}{\sqrt{\varepsilon_3}} \|w_x\| \|w\|) \\ & \leq C_4 (\varepsilon_5 \|w\|^2 + \frac{1}{\varepsilon_4} \|w_x\|^2) \end{aligned}$$

Collecting the estimates for different terms we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |w|_\alpha^2 &\leq -\|w_x\|^2 - \gamma \|w\|^2 \\ +C(\varepsilon_1 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2) + C_1(\varepsilon_2 \|w\|^2 + \frac{1}{\varepsilon_2} \|w_x\|^2) + C_5(\varepsilon_5 \|w\|^2 + \frac{1}{\varepsilon_4} \|w_x\|^2) \\ &\leq -\frac{1}{2} \|w_x\|^2 + C_0 \|w\|^2 \end{aligned}$$

where on the last step we have chosen  $C/\varepsilon_1 + C_1/\varepsilon_2 + C_5/\varepsilon_4 < 1/2$ . We rewrite our last result as

$$\frac{d}{dt} \|w\|^2 + \|w_x\|^2 \leq C \|w\|^2. \quad (7.4)$$

From this inequality we get first that  $\frac{d}{dt} \|w\|^2 \leq C \|w\|^2$  yielding, by Gronwall's inequality, that

$$\|w\|^2 \leq \|w_0\|^2 \exp(Ct);$$

at the same time by rearranging and integrating (7.4) we obtain

$$\int_0^t \|w_x\|^2 d\tau \leq \int_0^t (C \|w\|^2 - \frac{d}{dt} \|w\|^2) d\tau \leq \|w_0\|^2 \exp(Ct).$$

□

If the initial data are in  $H^1 \cap C_\alpha$  then a similar argument yields the following estimate analogous to (7.1):

$$\|U(t) - W(t)\|_1 \leq e^{Ct} \|U_0 - W_0\|_1 \quad (7.5)$$

Next we prove the differentiability in  $H^1$  that is sufficient for the validity of the dimension estimate because it implies the differentiability on  $\mathcal{A} \subset H^1$ .

**Theorem 7.2.** *Let  $U$  and  $W$  be two orbits  $U = T(t)U_0$ ,  $W = T(t)W_0$ ,  $U_0, W_0 \in H^1 \cap C_\alpha$ . Then there exists  $z(t)$  such that*

$$\|U(t) - W(t) - z(t)\| \leq \text{const} \|U_0 - W_0\|_1^2$$

as  $W_0 \rightarrow U_0$ .

In this case the Frechét differential of  $T(t)$  at the point  $U_0$  is the mapping  $z(0) = U_0 - W_0 \rightarrow z(t)$ , where  $z(t)$  solves the linearized problem.

*Proof.* The goal of the proof is to evaluate the difference between  $w = U - W$  and its approximation by the differential. We define  $z(x, t)$  as a solution of the free-interface problem linearized about the orbit  $U(x, t)$ :

$$\begin{aligned} z_t &= z_{xx} + [z_x(0, t)]U_x + [U_x(0, t)]z_x - \gamma z, \\ z(0, t) &= (g^{-1})'([U_x(0, t)])[z_x(0, t)], \quad z(x, 0) = U_0(x) - W_0(x), \end{aligned} \quad (7.6)$$

(see Theorem 6.1). For the difference  $y = w - z$  we have the following equations

$$\begin{aligned} y_t &= y_{xx} + [y_x(0, t)]U_x + [U_x(0, t)]y_x - [w_x(0, t)]w_x - \gamma y, \\ y(0, t) &= (g^{-1})'([U_x(0, t)])[y_x(0, t)] + (g^{-1})''(\theta)[w_x(0, t)]^2/2, \quad y(x, 0) = 0, \end{aligned}$$

We multiply the equation throughout by  $y$  and integrate to obtain the following identity for the  $H$  norm:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|y\|^2 &= \int_{R^\pm} y_{xx} y dx + [U_x(0, t)] \int_{R^\pm} y_x y dx + [y_x(0, t)] \int_{R^\pm} U_x y dx \\
&\quad - \gamma \int_{R^\pm} |y|^2 dx - \int_{R^\pm} [w_x(0, t)] w_x y dx \\
&= -[y_x] y|_0 - \|y_x\|^2 + [U_x(0, t)] \int_{R^\pm} y_x y dx + [y_x(0, t)] \int_{R^\pm} U_x y dx \\
&\quad - \gamma \int_{R^\pm} |y|^2 dx - \int_{R^\pm} [w_x(0, t)] w_x y dx
\end{aligned} \tag{7.7}$$

We need to estimate different terms in (7.7)

$$\begin{aligned}
|[y_x(0, t)] y(0, t)| &\leq C y^2(0, t) + C [w_x(0, t)]^2 |y(0, t)| \leq \frac{3}{2} C y^2(0, t) + \frac{1}{2} C [w_x(0, t)]^4 \\
&= B_1 y^2(0, t) + B_2 [w_x(0, t)]^4 \\
&\leq \frac{B_1}{2} \left| \int_{-\infty}^0 y_x^2 dx \right| + \frac{B_1}{2} \left| \int_0^\infty y_x^2 dx \right| + B_3 (\varepsilon_1 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2)^2 \\
&\leq B_1 (\varepsilon_1 \|y\|^2 + \frac{1}{\varepsilon_1} \|y_x\|^2) + B_3 (\varepsilon_1 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2)^2
\end{aligned}$$

Next,

$$\left| [U_x(0, t)] \int_{R^\pm} y_x y dx \right| \leq |[U_x(0, t)]| (\varepsilon_2 \|y\|^2 + \frac{1}{\varepsilon_2} \|y_x\|^2) \leq C_1 (\varepsilon_2 \|y\|^2 + \frac{1}{\varepsilon_2} \|y_x\|^2)$$

Also,

$$\begin{aligned}
&\left| [y_x(0, t)] \int_{R^\pm} U_x y dx \right| \leq (C_3 |y(0, t)| + B_4 [w_x(0, t)]^2) \\
&\quad \times \int_{R^\pm} |U_x y| dx \leq (C_3 |y(0, t)| + B_4 [w_x(0, t)]^2) \|y\| \|U_x\| \\
&\leq C_4 (\varepsilon_3 \|y\|^2 + \frac{1}{\varepsilon_3} \|y_x\|^2)^{1/2} \|y\| + B_5 (\varepsilon_1 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2 + \|w\|^2) \|y\| \\
&\leq C_4 (\sqrt{\varepsilon_3} \|y\| + \frac{1}{\sqrt{\varepsilon_3}} \|y_x\|) \|y\| + B_5 (\varepsilon_4 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2) \|y\| \\
&\leq C_5 (\varepsilon_5 \|y\|^2 + \frac{1}{\sqrt{\varepsilon_3}} \|y_x\|^2) + B_6 (\|w_x\|^4)
\end{aligned}$$

where the constants  $C_4$  and  $B_5$  include the factor  $\|U_x\|$ .

Finally,

$$\begin{aligned}
&\left| \int_{R^\pm} w_x [w_x(0, t)] y dx \right| \leq |[w_x(0, t)]| \|w_x\| \|y\| \\
&= \frac{\varepsilon}{2} \|w_x\|^2 |[w_x(0, t)]|^2 + \frac{1}{2\varepsilon} \|y\|^2 \leq C \frac{\varepsilon}{2} \|w_x\|^2 (\varepsilon_1 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2) + \frac{1}{2\varepsilon} \|y\|^2
\end{aligned}$$

Here we used the inequality in (7.3) to estimate  $[w_x(0, t)]^2$ .

Collecting the estimates for different terms we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y\|^2 &\leq -\gamma \|y\|^2 \\ &- \|y_x\|^2 + B_1(\varepsilon_1 \|y\|^2 + \frac{1}{\varepsilon_1} \|y_x\|^2) + B_3(\varepsilon_1 \|w\|^2 + \frac{1}{\varepsilon_1} \|w_x\|^2)^2 \\ &+ C_1(\varepsilon_2 \|y\|^2 + \frac{1}{\varepsilon_2} \|y_x\|^2) + C_5(\varepsilon_5 \|y\|^2 + \frac{1}{\sqrt{\varepsilon_3}} \|y_x\|^2) + B_6(\|w_x\|^4) \end{aligned}$$

Now we can select  $\varepsilon_1$ ,  $\varepsilon_2$ , and  $\varepsilon_3$  sufficiently large so that the coefficient by  $\|y_x\|^2$  is less than  $-1/2$ . We collect the like terms in the above inequality to obtain

$$\frac{1}{2} \frac{d}{dt} \|y\|^2 \leq -\frac{1}{2} \|y_x\|^2 + C \|y\|^2 + B \|w\|_1^4$$

We rewrite our last result as

$$\frac{d}{dt} \|y\|^2 + \|y_x\|^2 \leq C \|y\|^2 + B \|w\|_1^4 \quad (7.8)$$

from where it is clear that

$$\frac{d}{dt} \|y\|^2 \leq C \|y\|^2 + B \|w\|_1^4. \quad (7.9)$$

By Gronwall's inequality it yields

$$\|y\|^2 \leq B \exp(Ct) \int_0^t \|w\|_1^4 \exp(-C\tau) d\tau \leq B_7 \exp(Ct) \|w_0\|_1^4$$

In the above estimate we utilized (7.5). □

Finally (see [24]), the estimate for the dimension of the linear volume element and differentiability of the semigroup yield the estimate for the Hausdorff dimension of the attractor:

**Theorem 7.3.** *The Hausdorff dimension of the attractor  $\mathcal{A}$  is no larger than*

$$M = [(\frac{2\nu_0 + 1}{4\nu_0^2})^2 + \frac{1}{2}\mathcal{N}^2]/\gamma$$

cf. (6.11).

In conclusion it is worth mentioning that the estimate exhibits a transparent and physically natural dependence of the dimension on the heat loss and characteristics of the kinetics which are the defining factors of the dynamics. We note however that numerical simulations [14] on (1.1)-(1.4) show that the behavior without heat losses and with sufficiently low heat losses are qualitatively identical and exhibit the same variety of complex dynamical patterns.

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