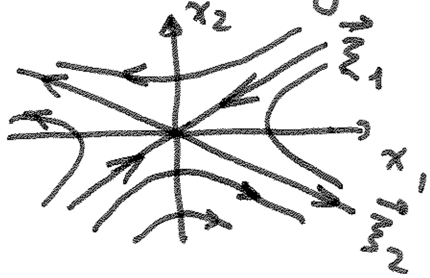
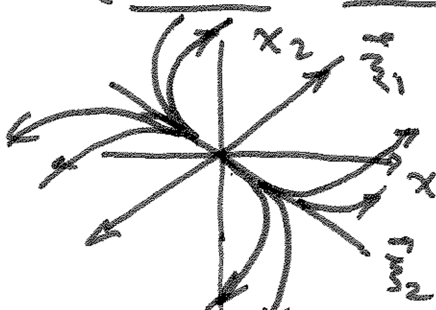
Q: What do we know about 2×2 <u>linear</u> systems? $\frac{d\vec{x}}{dt} = A\vec{x}$

A: Everything is determined by eigenvalues of A.

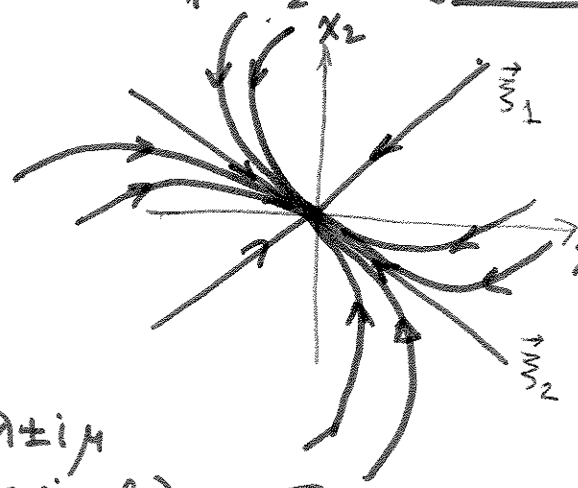
① Real eigenvalues of different signs $r_1 < 0 < r_2$
<u>Saddle</u>



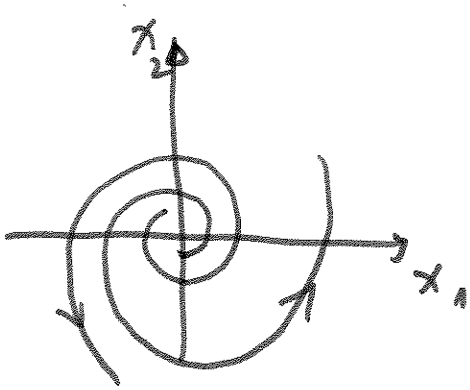
② Positive eigenvalues (<u>nodal source</u>) $r_1 > r_2 > 0$

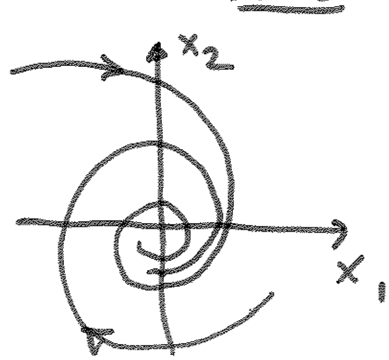


③ negative eigenvalues $r_1 < r_2 < 0$ (<u>nodal sink</u>)

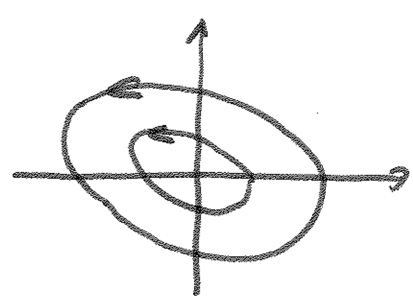


④ Complex eigenvalues $\lambda \pm i\mu$
$\lambda > 0$. (<u>spiral source</u>) $\lambda < 0$ (<u>spiral sink</u>)





⑤ Pure imaginary
$\lambda = \pm i\mu$ (<u>center</u>)



Q: Where can we study <u>nonlinear systems</u>?

A: We can study nonlinear systems if they behave like <u>almost</u> <u>linear</u>.

A system $\vec{x}' = A\vec{x} + \vec{g}(x)$ is <u>almost linear</u> in the neighborhood of the critical point 0 if $g(x)$ is "substantially smaller" than $\vec{x}$ (say, on the order of $\|\vec{x}\|^2$). [$\|\vec{g}(\vec{x})\|/\|\vec{x}\| \to 0$ as $\|\vec{x}\| \to 0$].

<u>Example</u> $\begin{cases} x' = x(1-x-y) \\ y' = y(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x) \end{cases} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x(x+y) \\ -\frac{y}{4}(3x+y) \end{bmatrix}$

$\vec{g} \to 0$ faster than $\|\vec{x}\|$ at 0.

Q: How do we study nonlinear systems?

A(short): Find critical points and determine <u>linear</u> <u>Behavior</u> about them.

Linearization about a critical point.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} F(x,y) \\ G(x,y) \end{matrix}$$

x_0, y_0 is a critical point, i.e $F(x_0,y_0)=0$
$G(x_0,y_0)=0$.

For our example, critical points are
$(0,0)$, $(1,0)$, $(0,2)$, $(\frac{1}{2},\frac{1}{2})$

Let's take $(1,0)$. To see what happens about it, introduce the "deviation" from $(1,0)$, $u=x-1$, $v=y-0$,
i.e $x=1+u$, $y=v$

$$(1+u)' = (1+u)(1-(1+u)-y) = -y-u+u(-u-y)$$

$$y' = y\left(\frac{1}{2}-\frac{1}{4}y-\frac{3}{4}(1+u)\right) = -\frac{1}{4}y-\left(\frac{1}{4}y^2-\frac{3}{4}uy\right)$$

$$\begin{pmatrix} u \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} u \\ y \end{pmatrix} + g$$

In general,

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = J \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + g$$

where $J = \begin{bmatrix} \frac{\partial F}{\partial x}(x_0,y_0) & \frac{\partial F}{\partial y}(x_0,y_0) \\ \frac{\partial G}{\partial x}(x_0,y_0) & \frac{\partial G}{\partial y}(x_0,y_0) \end{bmatrix}$ (Jacobian matrix).

<u>Stability</u> A critical point $\vec{x}_0$ is <u>stable</u> if solutions with initial data nearby <u>do not go away from</u> $\vec{x}_0$. It is <u>asymptotically stable</u> if they <u>approach $\vec{x}_0$ as $t \to \infty$</u>. (see pp. 472-3 for precise definition).

<u>Result</u>: Except for the case of <u>center</u>, nonlinear systems have exactly the same stability properties as their linearizations at critical points (Table 9.3.1)