Intro. Diff. Equations MATH-2400 - Exam #4 - 2001

1(a)(12p.). Find the general solution of the system:

$$\frac{dx}{dt} = 3x - 2y, \quad \frac{dy}{dt} = 2x - 2y;$$

1(b)(BONUS 2p.) Describe initial conditions (x_0, y_0) at $t = 0$, for which the solution $x(t), y(t)$ has a finite limit as $t \to \infty$.

Answer: **(a)** Write the system as

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}, \quad \text{then } \mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \exp(-\mathbf{t}) + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \exp(\mathbf{2t})$$

(b) The solution will have a finite limit as $t \to \infty$ if it contains only the decaying exponential, i.e. if $C_2 = 0$. Then $x_0 = C_1, y_0 = 2C_1$.

2(16p.). Consider systems of the type $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 constant matrix and $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. For each of the 4 cases below, the first column gives some information about the system or its solutions.

Answer the question in the second column.

Given Information	
(a) Eigenvalues: $\pm 5i$, $(x_1^0, x_2^0) = (1, 1)$	Sketch a possible phase trajectory in the x_1x_2-plane that starts at (x_1^0, x_2^0). How much time does it take to conclude the full cycle?
(b) $x_1(0) = 0$, $x_2(0) = -1$ Sketch of $x_1(t)$:	Sketch a possible phase portrait
(c) General solution: $\mathbf{x}(t) = c_1 \exp(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \exp(-t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$	Sketch a possible phase portrait

3.(a)(6p.) Reduce the second order equation $y'' + 2y' \sin t + 3y = 0$ to a system of first order differential equations.

Answer: (a)

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -3 & -2\sin t \end{bmatrix} \mathbf{x}, \quad \text{where } x_1 = y, x_2 = y'$$

(b)(4p.) Are vectors **a**, **b** eigenvectors of matrix **A**? If the answer is "yes", what are the corresponding eigenvalues?

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$$

Answer: (b) Apply the matrix **A** to **a** to obtain $[7, 6, -7]$ and to **b** to obtain $[0, 6, -6] = 3\mathbf{b}$. Thus **b** is an eigenvector with eigenvalue 3.

4(10p.). Find the solution of the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Answer: (a) General solution

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 2\sin t \\ \sin t - \cos t \end{bmatrix} e^{-\mathbf{2t}} + C_2 \begin{bmatrix} 2\cos t \\ 1 - \sin t \end{bmatrix} e^{-\mathbf{2t}}$$

to get initial conditions take $C_2 = 1/2, C_1 = 5/2$.