Homework 2, Fundamentals of Geometry, Spring 2012
Due Friday, February 10th

Problems are 10 points each, be sure to do the last problem. All homeworks are weighted equally, regardless of how many problems they contain.

Important Note on Problems from Pressley: There is a Hint Section and a Solution Section at the back of the book which contains most all problems. They both may or may not be of value to you, depending on the problem. But the write ups are usually too brief to qualify for a correct solution on the homework. When forming the solutions to these problems, you need to do more than just copy the solutions. **Most importantly, begin by giving a one paragraph overview of the main insights or techniques in the proof.** Then try to make the proof crystal clear by filling in the details and explaining steps as needed. Include pictures where necessary. If you would prefer, you are welcome to ignore the solution in the book and write your own (in some cases, this is a better way to do things).

1. A general Bézier curve is obtained by repeating linear interpolation. For example, for the cubic case, we would start with four control points, $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ and then form three points on the first layer:

$$\mathbf{P}_0^1(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1, \quad \mathbf{P}_1^1(t) = (1-t)\mathbf{P}_1 + t\mathbf{P}_2, \quad \mathbf{P}_2^1(t) = (1-t)\mathbf{P}_2 + t\mathbf{P}_3$$

where the superscript indicates the first layer. We then form the second layer by using the points found in the first layer with

$$\mathbf{P}_0^2(t) = (1-t)\mathbf{P}_0^1(t) + t\mathbf{P}_1^1(t), \quad \mathbf{P}_1^2(t) = (1-t)\mathbf{P}_1^1(t) + t\mathbf{P}_2^1(t)$$

and finally the cubic Bézier curve, say $\mathbf{b}(t)$ is just a combination of the two points of the second layer

$$\mathbf{b}(t) = (1-t)\mathbf{P}_0^2(t) + t\mathbf{P}_1^2(t)$$

which sweeps out a curve for $0 \le t \le 1$

   (a) For some nice set of control points, draw a large picture of the construction. Label all of the intermediate layers of points shown above and the ratios $t : (1-t)$ on your picture.

   (b) Show that $\mathbf{b}(t) = (1-t)^3\mathbf{P}_0 + 3(1-t)^2t\mathbf{P}_1 + 3(1-t)t^2\mathbf{P}_2 + t^3\mathbf{P}_3$

   (c) Show that $\mathbf{b}'(t) = 3\left((1-t)^2(\mathbf{P}_1 - \mathbf{P}_0) + 2(1-t)t(\mathbf{P}_2 - \mathbf{P}_1) + t^2(\mathbf{P}_3 - \mathbf{P}_2)\right)$ (Hint: Don't expand. Use the product rules and chain rules and collect things multiplying $(1-t)^2$, $2(1-t)t$, and t^2.

   (d) Find a formula for $\mathbf{b}''(t)$ in a similar format to that in part c.

   (e) Find $\mathbf{b}(0)$, $\mathbf{b}(1)$, $\mathbf{b}'(0)$, $\mathbf{b}'(1)$, $\mathbf{b}''(0)$, $\mathbf{b}''(1)$ in terms of the control points and carefully sketch each on a picture of the control points. (You can use the picture in part a if it is big enough and you use a lot of colors, otherwise, make a new picture.)

   (f) For a planar Bézier curve find "nice" formulas for the signed curvatures $\kappa_s(0)$ and $\kappa_s(1)$ in terms of the control points.

   (g) Find geometrical meaning for the signed curvature $\kappa_s(0)$ that you found in part f. The interpretation should be in terms of the control points and various distances and angles and maybe some scalars. Also comment on how the control points determine the sign of the signed curvature. Illustrate all of this with a picture (or pictures) of the control polygon (the control points connected with line segments).

(continued)

2. Osculating Circles. (Hand in plots and explanations as needed for each of parts b,c,d.)

   (a) Do problem 2.2.6 in Pressley. (Illustrate profusely.)

   (b) In Maple, create a plot that puts a lot of osculating circles on a general logarithmic spiral. Choose whatever k works and try intervals of length 2π and 4π and maybe others. What happens if you don't plot the logarithmic spiral but just all the osculating circles? Why?

   (c) Repeat part b for the ellipse parameterized by $(3\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$

   (d) Repeat part b for some other curves of your choosing. (Caution: Avoid inflection points or you may get division by 0 errors.)

3. Evolutes (Hand in plots and explanations as needed for each of parts b,c,d.)

   (a) Do problem 2.2.7 (Illustrate profusely.)

   (b) Draw a picture of the cycloid and its evolute in Maple. Connect points on the evolute to the corresponding points on the original curve by straight lines in your picture for a variety of points. Discuss your picture.

   (c) Repeat part b for the ellipse parameterized by $(3\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$

   (d) Repeat part b for some curves of your choosing. (Caution: Avoid inflection points or you may get division by 0 errors.)

4. Involutes and evolutes.

   (a) Do problem 2.2.8

   (b) *"Evolutes and involutes are inverse processes."*
   Make this statement as precise as possible and prove it.

5. Rank the problems (or parts of problems) on this homework in order of "coolness", from most cool to least cool. Explain how you came up with your ranking.