Homework I, Fundamentals of Geometry, Spring 2012
Due Friday, February 3rd

1. An involute of a circle is given by

$$\gamma(t) = \mathbf{u}(t) - t \cdot \mathbf{u}'(t) \quad t > 0, \quad \text{where } \mathbf{u}(t) = (\cos(t), \sin(t))$$

   (a) Parametrize the involute of the circle by arc length.

   (b) Find the equation of the involute of the involute of the circle.

   (c) Use Maple (or some software) to draw a picture of the circle, the involute of the circle and the involute of the involute of the circle on the same graph.

   (d) Use Maple (or some software) to animate the strings in the constructions of the picture in the previous exercise.

2. The general Logarithmic Spiral is given by $\gamma(t) = (e^{kt}\cos(t), e^{kt}\sin(t)), \quad -\infty < t < \infty$

   (a) Let $\mathbf{r}_\phi(t)$ be the result of rotating $\gamma(t)$ about the origin through angle ϕ. Let $\mathbf{s}_\alpha(t)$ be the result of scaling $\gamma(t)$ by α. Show that for any ϕ there is an α so that $\mathbf{r}_\phi$ and $\mathbf{s}_\alpha$ trace out the same curve (but with possibly different parameters.)

   (b) Reparameterize the general logarithmic spiral by arc length.

   (c) Find the involute of the general logarithmic spiral.

3. In this exercise, we ask you to parameterize the ellipse $x^2/9 + y^2 = 1$ in different ways. The simplest way is to write $\gamma(t) = (3\cos(t), 3\sin(t))$ but t is not very meaningful in this context.

   (a) Parameterize the ellipse using polar coordinates. That is, find the scalar function r so that the parametric curve $\mathbf{p}(\theta) = r(\theta)\mathbf{u}(\theta)$ traces out the ellipse where $\mathbf{u}(\theta) = (\cos(\theta), \sin(\theta))$. (This is a classic parameterization that is motivated by the study of orbital motion. See your Calculus book if you need help and for more details.)

   (b) A rational quadratic parameterization is one that looks like

$$\mathbf{Q}(t) = \left(\frac{x_1(t)}{q(t)}, \frac{x_2(t)}{q(t)}\right)$$

   where x_1, x_2, q are each polynomials of degree 2 or less and least one of them is degree 2.

   Follow the steps below to find a rational quadratic parameterization of the ellipse. Draw pictures to illustrate the construction.

   - Fix $P = (0, 1)$ as the top point on the ellipse.
   - For each point $(t, 0)$ on the x-axis, find the parametric curve, $\mathbf{L}(s)$, for the line from $(t, 0)$ to P.
   - Solve for where $\mathbf{L}(t)$ intersects the ellipse. One of the points will be P but there will be another point of intersection as well.
   - Use your solution to the previous part to write a rational quadratic parameterization of the ellipse.

   (c) Discuss the differences, advantages and disadvantages of each of the above two parameterizations.

4. The Tschirnhausen Cubic is given by $\mathbf{r}(t) = (3t^2, t^3 - 3t), \quad -\infty < t < \infty.$

   (a) Sketch the Tschirnhausen cubic and draw arrows to indicate its direction.

   (b) Find a formula for the involute of the Tschirnhausen cubic. Write a parameterization so that each coordinate is rational. (You need not multiple the polynomials out or expand.) What is the largest degree?