

Answers to Selected Problems on Exam 2, Math Analysis I, Fall 2005

1. Show that the union of a finite number of compact subsets of a metric space is compact.

Proof: Let the compact sets be U_1, U_2, \dots, U_n . Any open cover of the union of these sets is an open cover of each one and since each one is compact there is a finite subcover of the original open cover for each U_i . The union of these finite subcovers is a finite subcover for the union of the sets.

2. (a) Suppose that $f : (0, \infty) \rightarrow \mathfrak{R}$ is uniformly continuous on $(0, \infty)$ and x_1, x_2, \dots is a sequence of positive real numbers that converges to 0. Prove that $f(x_1), f(x_2), \dots$ is a Cauchy sequence.

Proof: Let $\epsilon > 0$ be given. By the uniform continuity of f , we may choose $\delta > 0$ so that for all $x, y > 0$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Since x_1, x_2, \dots is a Cauchy sequence, we can pick N so for all $n, m > N$ we have $|x_n - x_m| < \delta$. This implies that for all $n, m > N$, we have $|f(x_n) - f(x_m)| < \epsilon$. Thus $f(x_1), f(x_2), \dots$ is a Cauchy sequence.

- (b) Give an example where the statement in part a is false if the assumption that f is uniformly continuous is replaced by the weaker assumption that f is continuous.

Example Set $f(x) = 1/x$ and let the sequence of x_n 's be $1, 1/2, 1/3, 1/4, \dots$ which is Cauchy because it converges. But applying f yields $1, 2, 3, 4, 5, \dots$ which is not Cauchy.

3. Suppose $f : \mathfrak{R} \rightarrow \mathfrak{R}$ and $g : \mathfrak{R} \rightarrow \mathfrak{R}$ are both uniformly continuous and are both bounded functions on \mathfrak{R} . Prove that the product fg is uniformly continuous on \mathfrak{R} .

Proof: By the assumption of bounded functions, pick M_1, M_2 so that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in \mathfrak{R}$. Let $\epsilon > 0$ be given and by the assumption of uniform continuity pick $\delta_1 > 0$ so that $|x - y| < \delta_1$ implies that $|f(x) - f(y)| < \frac{\epsilon}{2M_2}$ and pick $\delta_2 > 0$ so that $|x - y| < \delta_2$ implies that $|g(x) - g(y)| < \frac{\epsilon}{2M_1}$. Let $\delta = \min(\delta_1, \delta_2)$. If $|x - y| < \delta$ then

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &< M_1 \frac{\epsilon}{2M_1} + M_2 \frac{\epsilon}{2M_2} = \epsilon. \end{aligned}$$

4. Let (E, d) be a metric space and for $S \subset E$ define \bar{S} , to be the intersection of all closed sets that contain S . Let $p \in E$. Using the given definition of \bar{S} , prove that $p \in \bar{S}$ if and only if every open ball in E with center p contains points of S .

Proof: To prove the forward direction, let $p \in \bar{S}$ and let $B_r(p)$ be an open ball with center p . Proceeding by contradiction assume that $B_r(p)$ contains no points of S . Then the complement of $B_r(p)$ is a closed set containing S and hence $\bar{S} \subset \mathcal{C}B_r(p)$ which contradicts the assumption that p was in \bar{S} . Hence $B_r(p)$ must contain points of S .

For the reverse direction, we will suppose p is not in \bar{S} and prove that there exists an open ball of center p that contains no points of S . Since \bar{S} is an intersection of closed sets, it is closed and hence the complement, $\mathcal{C}\bar{S}$, is open. Since p is not in \bar{S} we have $p \in \mathcal{C}\bar{S}$ and since this set is open, we may choose an open ball $B_r(p)$ of center p which is a subset of $\mathcal{C}\bar{S}$ and since $S \subset \bar{S}$ we have $B_r(p)$ contains no points of S and the proof is complete.

5. Suppose that S is a connected subset of a metric space (E, d) . Prove that the closure of S is connected. (You may use the definition of \bar{S} on the previous page of the exam.)

Proof: Recall that a set S is connected if it cannot be written as the disjoint union of two non-empty open subsets, where these sets are open **relative** to S . Also, as in one of the class discussion sessions, since $S \subset \bar{S}$, if a set $A \subset \bar{S}$ is open relative to \bar{S} then $A \cap S$ is open relative to S and that there exists a set A^* that is open relative to E so that $A^* \cap \bar{S} = A$.

To do the proof, we will suppose \bar{S} is not connected and show that this implies that S is not connected. So we pick non-empty sets A and B that are open in \bar{S} such that $A \cap B = \emptyset$ and $A \cup B = \bar{S}$. Set

$A' = A \cap S$ and $B' = B \cap S$ and note that A' and B' are open relative to S and that $A' \cap B' = \emptyset$ and $A' \cup B' = S$. It remains to show that A' and B' are non empty. We argue that A' is nonempty by contradiction. If A' is empty then A is open in \bar{S} and contains no points of S . Thus if we choose an A^* open in E such that $A^* \cap \bar{S} = A$ we have A^* is an open set in E that contains no points of S and hence its complement is a closed set in E which contains S which means its complement contains \bar{S} which implies that $A = A^* \cap \bar{S} = \emptyset$ contradicting the original assumption that A is non empty. Thus A' is non-empty and a similar argument shows that B' is non-empty.

6. Let $E = \mathfrak{R} \times \mathfrak{R}$. Define $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ and $d'((x_1, x_2), (y_1, y_2)) = 3|x_1 - y_1| + |x_2 - y_2|$. Let S be a subset of E . Prove that S is open in (E, d) if and only if S is open in (E, d') .

Hint of Proof: First prove that there exists positive constants C_1 and C_2 such that

$$C_1 d((x_1, x_2), (y_1, y_2)) \leq d'((x_1, x_2), (y_1, y_2))$$

and

$$C_2 d'((x_1, x_2), (y_1, y_2)) \leq d((x_1, x_2), (y_1, y_2))$$

for all $(x_1, x_2), (y_1, y_2)$ in E . Then show that for every open ball $B_r^{(E, d')}(p)$ in (E, d') there exist a radius q so the open ball $B_q^{(E, d)}(p)$ in (E, d) is a subset of $B_r^{(E, d')}(p)$ and vice-versa.

7. Suppose that $f : \mathfrak{R} \rightarrow \mathfrak{R}$ and $g : \mathfrak{R} \rightarrow \mathfrak{R}$ are continuous functions. If S is a dense subset of \mathfrak{R} and $f(x) = g(x)$ for all $x \in S$ then prove that $f(x) = g(x)$ for all $x \in \mathfrak{R}$.

Proof: A set S is dense in \mathfrak{R} if its closure $\bar{S} = \mathfrak{R}$. By a previous problem, this means that for all $y \in \mathfrak{R}$ every open ball centered at y contains points of S .

Proceeding by contradiction, suppose there exists y such that $f(y) \neq g(y)$ and set $\epsilon = |f(y) - g(y)|$ so $\epsilon > 0$. By the continuity of f and g at y pick $\delta_1 > 0$ so that $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \epsilon/2$ and pick $\delta_2 > 0$ so that $|x - y| < \delta_2$ implies $|g(x) - g(y)| < \epsilon/2$. Set $\delta = \min(\delta_1, \delta_2)$. Then since S is dense in \mathfrak{R} the open ball centered at y of radius δ contains points of S so we may pick an $x \in S$ such that $|x - y| < \delta$. We have $f(x) = g(x)$ so

$$|f(y) - g(y)| = |f(y) - f(x) + g(x) - g(y)| \leq |f(y) - f(x)| + |g(y) - g(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This contradicts the assumption that $|f(y) - g(y)| = \epsilon$.

8. Suppose that S is a compact subset of \mathfrak{R} and $f : S \rightarrow \mathfrak{R}$ is continuous.

- (a) Prove that f maps closed subsets of S to closed sets in \mathfrak{R} .
 (b) If additionally, f is 1-1 on S , prove that the inverse function, f^{-1} , is continuous.

Proof: See class notes corresponding to section 5.3.3 in Strichartz.