PHYS6530 *Quantum Mechanics III*

Fall 2005 Problem Set #1

Due at Start of Morning Class on September 14

1. *Quantum Mechanics Review Exercises.* This problem reviews material with which you should be familiar before taking this course. In all of these, let q and p represent quantum mechanical observables, which are conjugate variables. That is $[q, p] = i\hbar$.

   a. Given the eigenvector equation $q|q'\rangle = q'|q'\rangle$, and the "translation operator" $\mathcal{T}(dq') \equiv 1 - i(p/\hbar)dq'$, show that $\mathcal{T}(dq')|q'\rangle$ is also an eigenvector of q and determine its eigenvalue. Of course, neglect terms of $\mathcal{O}(dq'^2)$.

   b. By appropriately inserting a complete set of basis states into $\mathcal{T}(dq')|\alpha\rangle$, where $|\alpha\rangle$ is an arbitrary state vector, derive the action of p in q-space, namely

   $$p|\alpha\rangle = \int dq'|q'\rangle \left(-i\hbar\frac{\partial}{\partial q'}\langle q'|\alpha\rangle\right)$$

   c. Show that $\langle q'|p'\rangle = \exp(ip'q'/\hbar)$ *(This is a very important result that we will use to derive the path integral formulation of quantum mechanics. See Zee, Page 10.)*

   d. Put $\hbar \equiv 1$ so that $[q, p] = i$. For the "harmonic oscillator" Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$, find a (non-Hermitian) operator a such that $H = a^\dagger a + \frac{1}{2}$. Show that the eigenvalues of $N \equiv a^\dagger a$ are non-negative integers n, and determine the action of a and $a^\dagger$ on the eigenvectors $|n\rangle$.

2. *Calculus-of-Variations Review Exercises.* You should also be familiar with these concepts, which are the basis of the Lagrangian formulation of classical mechanics.

   a. Consider the function $f(x, y, y')$ where $y = y(x)$ is a differentiable function of the independent variable x. Show that in order to minimize the integral

   $$I[f] = \int_{x_A}^{x_B} f(x, y, y')dx$$

   where x_A, x_B, $y(x_A) = y_A$, and $y(x_B) = y_B$ are all fixed, then f must satisfy

   $$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

   (This is the Euler-Lagrange equation.) Try to prove this rigorously[‡], at least for the class of one-parameter functions $\bar{y}(x, \epsilon)$ with $\bar{y}(x, 0) = y(x)$, and $\bar{y}(x_A, \epsilon) = y_A$ and $\bar{y}(x_B, \epsilon) = y_B$ for all ϵ, and then considering $I(\epsilon)$.

[‡]Throughout this course, "rigorous" means rigorous for a physicist, but not for a mathematician.

b. Show that for the special case when $f(x, y, y') = f(y, y')$ doesn't depend explicitly on x, then the Euler-Lagrange equation can be integrated to give

$$y' \frac{\partial f}{\partial y'} - f = \text{constant}$$

c. Given two points A and B in a vertical xy plane, find the path $y(x)$ that a particle will travel in the shortest time. Assume that the particle travels downward from A to B under the influence of gravity alone. This famous example is called the brachistochrone problem. The shape $y(x)$ is called a "cycloid".

3. Consider the action (phase) integral for a particle of mass m moving in one dimension:

$$S[x(t)] = \frac{1}{\hbar} \int_0^T dt \frac{1}{2} m\dot{x}^2$$

The particle moves between $(x, t) = (0, 0)$ and $(d, T) = (1 \text{ cm}, 1 \text{ sec})$ along "path" #1, $x(t) = vt$, where $v = 1$ cm/sec; or "path" #2, $x(t) = \frac{1}{2}at^2$, where $a = 2$ cm/sec^2.

a. Compute the difference between S along each of the two paths. Which is smaller? Why did you know which would be smaller before you carried out the calculation?

b. Determine numerically this difference for a grain of sand ($m \sim 1$ mg) and for an electron ($mc^2 = 0.511$ MeV). Explain what this means for the range of applicability of quantum mechanics.

4. (Zee, Ex. I.2.1.) Verify the path integral equation for a Hamiltonian $H = p^2/2m + V(q)$:

$$\langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t) e^{i \int_0^T dt \left[\frac{1}{2} m\dot{q}^2 - V(q)\right]}$$

5. This problem attempts to give you some insight as to the nature of the transition to a continuum for a classical system. The point is that a one-dimensional collection of identical masses connected by "springs" has a finite number of fundamental frequencies, whereas a continuous string has an infinite number. Consider an actual string made of atoms spaced $a = 10^{-8}$ cm apart. Suppose the length of the string is 1 m, and it is kept at such a tension that the fundamental frequency is 100 cycles per second (Hz). Find the maximum, i.e. "cutoff", fundamental frequency. To what region of the visible spectrum does this correspond?