PHYS6530 *Quantum Mechanics III*

Fall 2004 Problem Set #1

Due at Start of Morning Class on September 15

1. *Quantum Mechanics Review Exercises.* This problem reviews material with which you should be familiar before taking this course. In all of these, let q and p represent quantum mechanical observables, which are conjugate variables. That is $[q,p]=i\hbar$.

   a. Given the eigenvector equation $q|q'\rangle = q'|q'\rangle$, and the "translation operator" $\mathcal{T}(dq') \equiv 1 - i(p/\hbar)dq'$, show that $\mathcal{T}(dq')|q'\rangle$ is also an eigenvector of q and determine its eigenvalue. Of course, neglect terms of $\mathcal{O}(dq'^2)$.

   b. By appropriately inserting a complete set of basis states into $\mathcal{T}(dq')|\alpha\rangle$, where $|\alpha\rangle$ is an arbitrary state vector, derive the action of p in q-space, namely
   $$p|\alpha\rangle = \int dq'|q'\rangle \left(-i\hbar\frac{\partial}{\partial q'}\langle q'|\alpha\rangle\right)$$

   c. Show that $\langle q'|p'\rangle = \exp(ip'q'/\hbar)$ *(This is a very important result that we will use to derive the path integral formulation of quantum mechanics. See Zee, Page 10.)*

   d. Put $\hbar \equiv 1$ so that $[q,p]=i$. For the "harmonic oscillator" Hamiltonian $H = \frac{1}{2}(p^2+q^2)$, find a (non-Hermitian) operator a such that $H = a^\dagger a + \frac{1}{2}$. Show that the eigenvalues of $N \equiv a^\dagger a$ are non-negative integers n, and determine the action of a and $a^\dagger$ on the eigenvectors $|n\rangle$.

2. (See Mandl & Shaw, Ex.1.1.) Consider the radiation field inside a cubic enclosure, as discussed in class. Any state $|n\rangle$ that contains a definite number of photons n has $\langle\vec{E}\rangle = 0$ so it cannot represent a classical field. Instead, consider the state defined as
   $$|c\rangle = \exp\left(-\frac{1}{2}|c|^2\right)\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}|n\rangle$$
   where c is a complex number. All photons have the same wave number and polarization.

   a. Show that $|c\rangle$ is normalized, i.e. $\langle c|c\rangle = 1$.

   b. Show that $|c\rangle$ is an eigenstate of the annihilation operator, where $a|c\rangle = c|c\rangle$

   c. Find the mean number of photons $\bar{N} = \langle c|N|c\rangle$ and also the root-mean-square $\sigma_N^2 = \langle c|(N-\bar{N})^2|c\rangle$, where $N = a^\dagger a$ is the photon number operator

   d. For the electric field operator $\vec{E}$ in the transverse gauge, determine the mean field $\langle c|\vec{E}|c\rangle$ and the root-mean-square field, in terms of the wave number $|\vec{k}|$, the cavity volume V, and the phase δ of c.

   The state $|c\rangle$ is called a "coherent" state. See the textbook by Loudon for a full discussion.

3. (Huang, Ex.1.1.) Consider an actual string made of atoms spaced $a = 10^{-8}$ cm apart. Suppose the length of the string is 1 m, and it is kept at such a tension that the fundamental frequency is 100 cycles per second (Hz). Find the cutoff frequency, and show that it lies in the infrared region of the spectrum. (This gives the Debye temperature.)

4. Consider the action (phase) integral for a particle of mass m moving in one dimension:

$$S[x(t)] = \frac{1}{\hbar}\int_0^T dt \frac{1}{2} m\dot{x}^2$$

The particle moves between $(x,t) = (0,0)$ and $(d,T) = (1 \text{ cm}, 1 \text{ sec})$ along "path" #1, $x(t) = vt$, where $v = 1$ cm/sec; or "path" #2, $x(t) = \frac{1}{2}at^2$, where $a = 2$ cm/sec^2.

   a. Compute the difference between S along each of the two paths. Which is smaller? Why did you know which would be smaller before you carried out the calculation?

   b. Determine numerically this difference for a grain of sand ($m \sim 1$ mg) and for an electron ($mc^2 = 0.511$ MeV). Explain what this means for the range of applicability of quantum mechanics.

5. (Zee, Ex. I.2.1.) Verify the path integral equation for a Hamiltonian $H = p^2/2m + V(q)$:

$$\langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t) e^{i\int_0^T dt\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]}$$