PHYS6520 *Quantum Mechanics II*

Spring 2003 Problem Set #6

Due on Monday, April 14, at start of class.

1. a. (Merzbacher Exercise 21.6.) Show that $\langle qr|V|ts\rangle = \langle rq|V|st\rangle$ where

$$\langle qr|V|ts\rangle = \sum_{ij} \langle L_q|K_i\rangle\langle K_i|L_t\rangle\langle L_r|K_j\rangle\langle K_j|L_s\rangle V_{ij}$$

where V_{ij} is a real, symmetric matrix.

   b. (Merzbacher Exercise 21.7.) Show that the square of an additive one-particle operator is generally expressible as the sum of an additive two-particle operator and an additive one-particle operator.

2. (See Merzbacher Exercise 21.10.) For either Bose-Einstein or Fermi-Dirac statistics, prove the following two relations, which we used to derive the equation of motion for the field operator $\Psi(\vec{x}, t)$:

$$\left[a_j, a_k^\dagger a_\ell\right] = a_\ell \delta_{kj}$$

and

$$\left[a_j, a_q^\dagger a_r^\dagger a_s a_t\right] = a_r^\dagger a_s a_t \delta_{qj} + a_q^\dagger a_t a_s \delta_{rj}$$

3. (Merzbacher Problem 21.1.)

   a. Show that if $V(r)$ is a two-particle interaction that depends only on the distance r between te particles, the matrix element of the interaction in the $\vec{k}$-representation may be reduced to

$$\langle \vec{k}_3\vec{k}_4|V|\vec{k}_1\vec{k}_2\rangle = \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)\frac{1}{(2\pi)^3}\int V(r)e^{-i\vec{q}\cdot\vec{x}}d^3x$$

where $\hbar\vec{q}$ is the momentum transfer $\hbar(\vec{k}_3 - \vec{k}_1)$.

   b. For this interaction, show that the mutual potential energy operator is

$$\mathcal{V} = \frac{1}{2}\int\int\int d^3k_1 d^3k_2 d^3q \Phi^\dagger(\vec{k}_1 + \vec{q})\Phi^\dagger(\vec{k}_2 - \vec{q})\Phi(\vec{k}_2)\Phi(\vec{k}_1)F(\vec{q})$$

where $F(\vec{q})$ is the Fourier transform of the displacement-invariant interaction.

4. (Merzbacher Problem 21.2.) Show that the diagonal part of the interaction operator $\mathcal{V}$, found in Problem 21.1 in the $\vec{k}$-representation, arises from momentum transfers $\vec{q} = 0$ and $\vec{q} = \vec{k}_2 - \vec{k}_1$, respectively. Write down the two interaction terms and identify them as *direct* ($\vec{q} = 0$) and *exchange* ($\vec{q} = \vec{k}_2 - \vec{k}_1$) interactions. Draw the corresponding diagrams, following Figure 21.1.

5. a. (See Merzbacher Exercise 23.1.) Using commutation relations, show that

$$U_R(\vec{k}) a_\pm^\dagger U_R^\dagger(\vec{k}) = e^{-i\alpha \vec{J}\cdot\hat{k}/\hbar} a_\pm^\dagger(\vec{k}) e^{+i\alpha \vec{J}\cdot\hat{k}/\hbar} = e^{\mp i\alpha} a_\pm^\dagger(\vec{k})$$

where

$$\vec{J}\cdot\hat{k} = \hbar\left[a_+^\dagger(\vec{k}) a_+(\vec{k}) - a_-^\dagger(\vec{k}) a_-(\vec{k})\right]$$

b. (See Merzbacher Exercise 23.2.) For two unit vectors $\hat{e}_{\vec{k}}^{(1)}$ and $\hat{e}_{\vec{k}}^{(2)}$ which are mutually perpendicular and also perpendicular to $\vec{k}$, calculate the Hermitian scalar and vector products of the unit vectors $\hat{e}_{\vec{k}}^{(+)}$, $\hat{e}_{\vec{k}}^{(-)}$, and $\hat{k}$, given the definitions

$$\begin{aligned} \hat{e}_{\vec{k}}^{(+)} &= -\frac{1}{\sqrt{2}}\left(\hat{e}_{\vec{k}}^{(1)} + i\hat{e}_{\vec{k}}^{(2)}\right) \\ \hat{e}_{\vec{k}}^{(-)} &= \frac{1}{\sqrt{2}}\left(\hat{e}_{\vec{k}}^{(1)} - i\hat{e}_{\vec{k}}^{(2)}\right) \end{aligned}$$

Note that for complex unit vectors, the Hermitian scalar and vector products are $\hat{n_1}^* \cdot \hat{n_2}$ and $\hat{n_1}^* \times \hat{n_2}$ respectively.

6. (Merzbacher Exercise 23.10.) Reproduce the derivation of the matrix element (23.45) for absorption or emission of a photon by a charged particle system.

7. (Merzbacher Exercise 23.12.) Using the hydrogenic wave functions (12.92), compute the spontaneous emission rate for the $2p \to 1s$ transition in the hydrogen atom. Evaluate its reciprocal, the mean lifetime of the $2p$ state.