# PHYS6520 *Quantum Mechanics II*

Spring 2003 Problem Set #4

Due at Start of Class on March 17 (First class after Spring Break)

You are to work independently on this problem set. You are free to use whatever notes, books, computers, or other reference works you feel are useful. You are also free to consult the course instructor or teaching assistant for help. You may *not*, however, consult other students in the class.

You may want to include some plots and graphical comparisons. You can use whatever plotting program you like. Typical programs used at RPI are MAPLE and MATLAB, or various UNIX utilities.

Please attach this page to your homework solutions along with your signature, below.

"I have complied with the requirement that I work independently on this problem set. I have not consulted with anyone other than the course instructor and teaching assistant in preparing these solutions."

Signature: ______________________________

Print name: ______________________________

1. (See Sakurai Problem 5.10.) Consider a spinless particle in a two-dimensional infinite square well:

$$V = \begin{cases} 0 & \text{for } 0 \le x \le a, 0 \le y \le a \\ \infty & \text{otherwise} \end{cases}$$

   a. What are the three lowest energy eigenvalues? Which, if any of these, correspond to degenerate states?

   b. A perturbation $V_1 = \lambda xy$, for $0 \le x \le a$ and $0 \le y \le a$, is added. Calculate the energy shift to lowest non-vanishing order for all states identified in (a). Draw an energy diagram showing how each energy level shifts or splits, when $\lambda$ changes from zero to some positive value.

2. (See Sakurai Problem 5.30.) Consider a two-level system with $E_1 < E_2$. There is a time-dependent potential that connects the two levels as follows

$$V_{11} = V_{22} = 0 \text{ and } V_{12} = \gamma e^{i\omega t} \text{ where } \gamma \text{ is real}$$

   At $t = 0$, it is known that only the lower level is populated. That is, $c_1(0) = 1$ and $c_2(0) = 0$.

   a. Show that you can find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ *exactly* by solving the coupled differential equation

$$i\hbar \dot{c_k} = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_{kn} t} c_n$$

   for $k = 1, 2$. Determine $|c_1(t)|^2$ and $|c_2(t)|^2$. (The answer is given in the textbook.)

   b. Do the same problem using time-dependent perturbation theory to lowest non-vanishing order. Compare the two approaches for small values of $\gamma$. Treat the following two cases separately: (i) $\omega$ very different from $\omega_{21}$, and (ii) $\omega$ very close to $\omega_{21}$.

3. A plane electromagnetic wave, covering a range of frequencies, is polarized in the $\hat{x}$ direction and travels in the $\hat{z}$ direction. It impinges on a particle with charge $e$ and mass $m$, bound in a one dimensional simple harmonic oscillator with fundamental frequency $\omega_0$. The harmonic oscillator is in its ground state $|0\rangle$, and oriented in the $\hat{x}$ direction.

   a. Calculate the integrated absorption cross section to an arbitrary final state $|n\rangle$.

   b. Explicitly perform the sum over all states $|n\rangle$ and compare the result to the TRK sum rule.

4. (Merzbacher Exercise 13.10.) Obtain the differential scattering cross section in the Born approximation for the potential

$$V(r) = -V_0 e^{-r/a} \qquad\qquad (V_0 > 0)$$

What is the criterion in this case for the approximation to be valid?

5. a. In class, we showed that the coordinate space representation of $|\psi^{(+)}\rangle$ satisfies

$$\psi^{(+)}(\vec{x}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{x}} + \frac{2m}{\hbar^2} \int d^3x' G_+(\vec{x},\vec{x}')V(\vec{x}')\psi^{(+)}(\vec{x}')$$

and that it also solves the time-independent Schrödinger equation for this system. Use these facts to demonstrate that the Green's function satisfies the inhomogeneous Helmholtz equation

$$(\vec{\nabla}^2 + k^2)G_+(\vec{x},\vec{x}') = \delta(\vec{x} - \vec{x}')$$

Note that the Laplacian only operates on the $\vec{x}$ coordinate, not $\vec{x}'$.

b. Show that the inhomogeneous Helmholtz equation is satisfied by

$$G_+(\vec{x},\vec{x}') = -\frac{1}{4\pi} \frac{\exp(ik|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}$$

6. (See Merzbacher Exercise 13.15.) Calculate and make a polar plot of the differential scattering cross section from a hard sphere for $ka = 1/5$, 1, and 5, using the first three partial waves, i.e. $\ell = 0$, 1, and 2. Integrate each to get the total cross section and comment on the results.

7. A particle is scattered by a spherically symmetric potential at sufficiently low energy that the phase shifts $\delta_\ell = 0$ for $\ell > 1$. (That is, only $\delta_0$ and $\delta_1$ are nonzero.) Show that the differential cross section has the form

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta + C\cos^2\theta$$

and determine $A$, $B$, and $C$ in terms of the phase shifts. Determine the total cross section in terms of $A$, $B$, and $C$.