PHYS6520 *Quantum Mechanics II*

Spring 2003 Problem Set #1

Due at Start of Class on January 27

1. Consider a two-state ($|1\rangle$ and $|2\rangle$) system with a Hamiltonian $H = H_0 + V$ represented by a 2×2 matrix using the $|1, 2\rangle$ basis, and $H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$ with $E_1 \neq E_2$.

   a. For $V = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix}$ show that the exact energy eigenvalues are given perfectly by first order perturbation theory.

   b. For $V = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$ show that first order perturbation theory fails miserably to predict the correct energy eigenvalues. Use an expansion of the exact solution to explain why this is expected.

   c. For V as given in (b), determine the energy eigenvalues using second order perturbation theory, and compare to the exact result.

2. Consider a perturbation $V = bx^4$ to the simple harmonic oscillator Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}$$

   a. Show that the first order energy shift is given by

$$E_n^{(1)} = \frac{3\hbar^2 b}{4m^2\omega^2}(1 + 2n + 2n^2)$$

   b. Argue that no matter how small b is, the perturbation expansion will break down for some sufficiently large n. What is the physical reason for this?

3. (See Sakurai 5.2.) In nondegenerate time-independent perturbation theory, what is the probability of finding in a perturbed energy eigenstate ($|n\rangle$) the corresponding unperturbed eigenstate ($|n^{(0)}\rangle$)? Solve this up to terms of order g^2 where g is the perturbation expansion parameter. Be aware of the normalization condition used in the perturbation expansion.

4. An electric field $\mathcal{E}$ applied to a simple harmonic oscillator leads to a perturbation $V(x) = -q\mathcal{E}x$ for a particle of charge q. Another way to look at this is to realize that the electric field gives rise to a displacement δ by balancing the spring force against the electric force. To first order in this displacement, show that the state given by $\mathcal{T}(\delta)|n^{(0)}\rangle$, where $\mathcal{T}$ is the displacement operator and $|n^{(0)}\rangle$ is the unperturbed oscillator state, is the same as the state given by first order perturbation theory.

5. (See Sakurai 5.4.) Consider an isotropic harmonic oscillator in *two* dimensions. The Hamiltonian is given by

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2\right)$$

   a. What are the energies of the three lowest-lying states? Is there any degeneracy?

   b. We now apply a perturbation

$$V(x, y) = \delta m\omega^2 xy$$

   where $\delta \ll 1$ is a dimensionless real number. Find the zeroth order energy eigenket and the corresponding energy to first order (that is, the unperturbed energy plus the first order energy shift) for each of the three lowest lying states.

   c. Solve the $H_0 + V$ problem *exactly*. Compare with the perturbation results.

6. (See Shankar 17.3.2.) The Hamiltonian for a spin-1 particle, with no orbital angular momentum components, in a certain crystal may be given by

$$H = \frac{A}{\hbar^2}S_z^2 + \frac{B}{\hbar^2}(S_x^2 - S_y^2)$$

where $\vec{S}$ is the spin angular momentum operator.

   a. Assuming $B \ll A$ and treating the second term as a perturbation, calculate the unperturbed energies and the first order correction. *Beware of degeneracies.*

   b. Solve the problem exactly and compare to the perturbation result.