PHYS6510 *Quantum Mechanics I*

Fall 2002 Problem Set #6

Due on Thursday November 21, at start of class.

1. (See Sakurai 3.8.) We have seen two different 2×2 representations of the rotation operator. Sakurai Eq.(3.2.45) is in terms of a normalized rotation axis vector $\hat{n}$ and a rotation angle ϕ. On the other hand, Sakurai Eq.(3.3.21) is in terms of Euler angles α, β, and γ. Derive ϕ in terms of α, β, and γ, and show that it reduces to the correct result for the two cases $\alpha = \gamma = 0$, and $\beta = 0$. (You may find it useful to use Sakurai Eq.(1.5.16b), a theorem about "traces", which we did not cover in class.)

2. (See Sakurai 3.18.) Consider an orbital angular momentum eigenstate $|\ell = 2, m = 0\rangle$. Suppose this state is rotated by an angle β around the y-axis. Find the probability for the new state to be found in $m = 0$, ± 1, and ± 2, and check that the sum of all of these probabilities is equal to one.

3. (See Sakurai 3.22.)

   a. Consider a system with $j = 1$. Explicitly write $\langle j = 1, m'|J_y|j = 1, m\rangle$ in 3×3 matrix form, that is, verify Sakurai Eq.(3.5.54).

   b. Show that for $j = 1$, you can replace $\exp(-iJ_y\beta/\hbar)$ by

   $$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1 - \cos\beta)$$

   c. Write $d^{(j=1)}(\beta)$ in 3×3 matrix form, that is, verify Sakurai Eq.(3.5.57).

4. (See Sakurai 3.3.) The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as

   $$H = A\vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right)$$

   Using the *coupled representation* in which $\vec{S}^2 = \left(\vec{S}^{(e^-)} + \vec{S}^{(e^+)}\right)^2$ and $S_z = S_z^{(e^-)} + S_z^{(e^+)}$ are diagonal, obtain the energy eigenvalues and eigenvectors and classify them according to the quantum numbers associated with constants of the motion.

5. Verify that the spin spherical harmonics, i.e. Sakurai (3.7.64)

   $$\mathcal{Y}_\ell^{j=\ell\pm 1/2,m} = \frac{1}{\sqrt{2\ell+1}}\begin{pmatrix} \pm\sqrt{\ell \pm m + \frac{1}{2}}Y_\ell^{m-1/2}(\theta,\phi) \\ \sqrt{\ell \mp m + \frac{1}{2}}Y_\ell^{m+1/2}(\theta,\phi) \end{pmatrix}$$

   are simultaneous eigenfunctions of $\vec{L}^2$, $\vec{S}^2$, $\vec{J}^2$, and $\vec{L} \cdot \vec{S}$, and derive their eigenvalues. You may want to use the relationship

   $$\vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L} \cdot \vec{S} = \vec{L}^2 + \vec{S}^2 + L_+S_- + L_-S_+ + 2L_zS_z$$

6. (See Sakurai 3.25.) Consider a spherical tensor of rank 1 (that is, a vector)

$$V_{\pm 1}^{(1)} = \mp \frac{V_x \pm iV_y}{\sqrt{2}}, \qquad V_0^{(1)} = V_z$$

Using the expression for $d^{(j=1)}(\beta)$ in 3×3 matrix form, evaluate

$$\sum_{q'} d_{qq'}^{(1)}(\beta) V_{q'}^{(1)}$$

and show that your results are just what you expect from the transformation properties of $V_{x,y,z}$ under rotations about the y-axis.

7. (See Sakurai 3.28.)

   a. Using spherical harmonics, write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.

   b. The expectation value

   $$Q \equiv e\langle \alpha, j, m = j | (3z^2 - r^2) | \alpha, j, m = j \rangle$$

   is known as the *quadrupole moment*. Using the Wigner-Eckart theorem, evaluate

   $$e\langle \alpha, j, m' | (x^2 - y^2) | \alpha, j, m = j \rangle$$

   in terms of Q and appropriate Clebsch-Gordan coefficients.