PHYS6510 *Quantum Mechanics I*

Fall 2002 Problem Set #5

Due at Start of Class on Monday, November 4

1. For an angular momentum state $|jm\rangle$, evaluate

   a. Expectation values $\langle J_x \rangle$ and $\langle J_y \rangle$.

   b. Expectation values $\langle J_x^2 \rangle$ and $\langle J_y^2 \rangle$.

   c. The uncertainty product $\langle(\Delta J_x)^2\rangle\langle(\Delta J_y)^2\rangle$ and compare it with the generalized uncertainty principle.

   d. The sum $\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle$ and show that it equals $\langle \vec{J}^2 \rangle$.

2. (See Sakurai 3.3.) The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as

$$H = A\vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right)$$

   Suppose the spin state of the system is given by the spinor product $\chi_+^{(e^-)}\chi_-^{(e^+)}$, and that the e^+ (e^-) operators only act on $\chi_+^{(e^+)}$ ($\chi_+^{(e^-)}$).

   a. Is this an eigenfunction of H in the limit that $A \to 0$ and $B \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H.

   b. Repeat part (a) in the limit that $A \neq 0$ and $B \to 0$.

3. The orbital angular momentum operator is $\vec{L} = \vec{x} \times \vec{p}$, in terms of the position operator $\vec{x}$ and the momentum operator $\vec{p}$. Using the commutation relations for $\vec{x}$ and $\vec{p}$, show that $\vec{L}$ indeed possesses the fundamental commutation relations for angular momentum

$$[L_x, L_y] = i\hbar L_z, \qquad [L_y, L_z] = i\hbar L_x, \qquad [L_z, L_x] = i\hbar L_y$$

4. (See Sakurai 3.15.) The wave function of a particle subjected to a spherically symmetric potential $V(r)$ is given by

$$\psi(\vec{x}) = (x + y + 3z)f(r)$$

   a. Is ψ an eigenfunction of $\vec{L}^2$? If so, what is the value of the quantum number ℓ? If not, what are the possible values of ℓ we may obtain when $\vec{L}^2$ is measured?

   b. What are the probabilities for the particle to be found in various states with quantum number m_ℓ?

   c. Suppose that $\psi(\vec{x})$ is an energy eigenfunction with eigenvalue E. How would you find the potential $V(r)$?

5. Using differential operators in position space, show that

   a. $L_- Y_{1,1}(\theta, \phi)$ is proportional to $Y_{1,0}(\theta, \phi)$ with the correct coefficient.

   b. $Y_{1,1}(\theta, \phi)$ is an eigenfunction of $\vec{L}^2$ with the correct eigenvalue.

6. An electron in the Coulomb field of the proton is in the state

$$|\alpha\rangle = \frac{4}{5}|1,0,0\rangle + \frac{3i}{5}|2,1,1\rangle$$

   where $|n, \ell, m\rangle$ are the usual energy eigenstates of hydrogen.

   a. What is the expectation value of the energy for this state? What are $\langle \vec{L}^2 \rangle$ and $\langle L_z \rangle$?

   b. What is $|\alpha, t\rangle$? Which of the expectation values in part (a) vary with time?

7. The electron neutrino ν_e is generally considered to be a massless particle. Direct measurements of its mass have been carried out by measuring the beta decay spectrum of tritium, ^{3}H, which decays to ^{3}He with the emission of a β^- particle (an electron) and the neutrino. A distortion in the shape of the β spectrum near the "endpoint" might indicate a nonzero neutrino mass, but one has to be careful of other physical processes which can distort the spectrum.

   a. Assume a tritium atom in its ground state. The nucleus changes instantaneously into a ^{3}He nucleus. Calculate the probability that the resulting ^{3}He ion is in its ground state. (Obtain a numerical answer.)

   b. Discuss the effect this would have on the beta decay spectrum, as it relates to a neutrino mass measurement. See V.M. Lobashev et al., Phys. Lett. B460(1999)227 and C. Weinheimer et al., Phys. Lett. B460(1999)219 for recent experimental measurements of tritium beta decay.