PHYS6510 *Quantum Mechanics I*

Fall 2002 Problem Set #2
Due at Start of Class on September 23

1. (See Sakurai 1.26.) Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the matrix elements of an operator with the general form

$$U = \sum_r \left| b^{(r)} \right\rangle \left\langle a^{(r)} \right|$$

2. (See Sakurai 1.30.)

   a. Show that the translation operator for a finite spatial displacement $\vec{\ell}$ is given by

   $$\mathcal{T}(\vec{\ell}) = \exp\left(-\frac{i\vec{p}\cdot\vec{\ell}}{\hbar} \right)$$

   where $\vec{p}$ is the momentum operator.

   b. Evaluate $\left[x_i, \mathcal{T}(\vec{\ell}) \right]$ where $x_{1,2,3} = x, y, z$.

   c. Using the result in (b) (or otherwise), demonstrate how the expectation value $\langle \vec{x} \rangle$ changes under translation.

3. (See Sakurai 1.11 and 2.2.)

   a. A two-state system is characterized by the Hamiltonian

   $$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}\left[|1\rangle\langle 2| + |2\rangle\langle 1|\right]$$

   where H_{11}, H_{22}, and H_{12} are real numbers with the dimensions of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$.

   b. Suppose instead that the typist made a mistake and wrote

   $$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2|$$

   What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using an illegal Hamiltonian of this kind. (You may assume $H_{11} = H_{22} = 0$ for simplicity.)

4. (Sakurai 2.5.) Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x)$$

By calculating $[[H, x], x]$ prove

$$\sum_{a'} |\langle a''|x|a'\rangle|^2 \left(E_{a'} - E_{a''}\right) = \frac{\hbar^2}{2m}$$

5. (See Sakurai 2.3.) An electron is subject to a uniform, time-independent magnetic field of strength B in the positive $z-$direction. At $t = 0$ the electron is known to be in an eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector lying in the xz-plane, that makes an angle β with the $z-$axis.

   a. Obtain the probability for finding the electron in the $|S_x; +\rangle$ state as a function of time.

   b. Find the expectation value of S_x as a function of time.

   c. Show that your answers make good sense for the cases $\beta \to 0$ and $\beta \to \pi/2$.

6. a. Show that if the wave function $\langle x'|\alpha\rangle$ is purely real, then the expectation value of momentum $\langle p\rangle$ for any state $|\alpha\rangle$ must be zero.

   b. In class, we showed that for a freely moving, Gaussian, one-dimensional wave packet

   $$\psi_\alpha(x', 0) \equiv \langle x'|\alpha\rangle = C \exp\left[-\frac{x'^2}{2d^2} + ikx'\right]$$

   we have $\langle x\rangle = 0$, $\langle x^2\rangle = d^2/2$, and so $\langle(\Delta x)^2\rangle = d^2/2$. For this wave packet, further calculate ...

   - the normalization constant C. *(You may assume C is real.)*
   - the expectation value of momentum, $\langle p\rangle$
   - the expectation value $\langle p^2\rangle$
   - the momentum uncertainty $\langle(\Delta p)^2\rangle$

   and show that the uncertainty product for x and p is in fact the minimum value allowed by the uncertainty principle.

   c. Determine the momentum space wave function $\phi_\alpha(p', 0) \equiv \langle p'|\alpha\rangle$ for the Gaussian wave packet of part (b).

   d. Using (c), show that, for $k = 0$ (i.e. a stationary wave packet), the time dependent wave function is given by

   $$\psi_\alpha(x', t) = C \frac{1}{\left[1 + i\frac{\hbar t}{md^2}\right]^{1/2}} \exp\left[-\frac{1}{2}\frac{x'^2}{d^2 + i\frac{\hbar t}{m}}\right]$$

   and so determine the uncertainty $\langle(\Delta x)^2\rangle$ at a time t.