PHYS6510 *Quantum Mechanics I*

Fall 2002 Problem Set #1

Due at Start of Class on September 9

1. Physics is based on making assumptions, and then deriving predictions from these assumptions that stand up to experimental observation. In this problem, you are to show that Maxwell's Equations (for example, Jackson, 3rd Ed., Eqs. 6.6) predict that electric charge is conserved.

   a. Show that Maxwell's Equations predict that

   $$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

   where, using SI units, $\vec{J}$ is the current density and ρ is the charge density. (This is called the Continuity Equation.)

   b. Consider the charge in some enclosed region of space, and the net flow of charge into and out of that space. Demonstrate that charge conservation follows directly from the Continuity Equation.

2. For operators A and B, and complex number λ, prove the Baker-Hausdorff lemma

   $$f(\lambda) \equiv \exp(\lambda A) B \exp(-\lambda A) = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \frac{\lambda^3}{3!}[A,[A,[A,B]]] + \cdots$$

   i.e. Sakurai Eq.(2.3.47), by considering a Taylor series of the function $f(\lambda)$. *(It is easiest to start by showing that* $df/d\lambda = [A, f(\lambda)]$*.)*

3. In class, we showed that the spin-1/2 operators S_x, S_y, and S_z could be constructed as outer products of the z-projection up and down states $|\pm\rangle$. (See Sakurai Equations (1.3.36), (1.4.18a), and (1.4.18b).) Using only the orthonormality of $|+\rangle$ and $|-\rangle$, prove that

   $$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k$$

   i.e. Sakurai Eq.(1.4.20), where ε_{ijk} is the totally antisymmetric symbol for which $\varepsilon_{123} = +1$ and for any cyclic permutation of indices, -1 for any two indices reversed, and 0 if any two indices are the same. *(Note: This result shows that these forms of the spin-1/2 operators satisfy the fundamental commutation relations of angular momentum, which we will study later.)*

4. (Sakurai 1.20.) Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

   $$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle$$

   Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

5. Starting with the column vector representation for the z spin-1/2 up and down states

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

use the outer product forms of S_x, S_y, and S_z to show that the spin-1/2 operator $\vec{S} \doteq (\hbar/2)\vec{\sigma}$ where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The σ_i are called Pauli Matrices, and we will use them often.

6. (See Sakurai Problem 1.9.) In class we derived the eigenstates $|S_x;\pm\rangle$ and $|S_y;\pm\rangle$ for the spin-1/2 operators S_x and S_y, in terms of the eigenstates $|\pm\rangle$ of the operator S_z. (Equations 1.4.17.) In this problem, we will work with an arbitrary projection of $\vec{S}$ onto the direction $\hat{n}$, where $\hat{n}$ is characterized by the angles α and β shown on page 62 of Sakurai.

   a. Construct the eigenstates $|\vec{S}\cdot\hat{n};\pm\rangle$ for the operator $\vec{S}\cdot\hat{n}$. Do this by solving the eigenvalue problem directly. *(The result for $|\vec{S}\cdot\hat{n};+\rangle$ is given in Sakurai.)*

   b. Show that your answer reduces to the correct result for $\hat{n} = \hat{x}$, $\hat{y}$, and $\hat{z}$. (If you encounter an arbitrary phase factor, indicate the value it must have to get the usual answer.)

   c. Suppose that a measurement of S_z is carried out on a particle in the state $|\vec{S}\cdot\hat{n};+\rangle$. What are the probabilities that this measurement yields the values $\pm\hbar/2$?

   d. Determine the uncertainty $\langle(\Delta S_z)^2\rangle^{1/2}$ for the measurement in (c). Argue that your expression gives the correct answer for $\beta = 0$ or π.