Working out Precession

(Sakurai Pages 76-78)
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This brief note works out some of the details on the precession of an electron in a constant magnetic field. The topic is discussed on pages 76-78 of Sakurai, *Modern Quantum Mechanics*.

Classically, the potential energy of a magnetic moment $\vec{\mu}$ in a magnetic field $\vec{B}$ is

$$U = -\vec{\mu} \cdot \vec{B}$$

as mentioned in Sakurai on page 3. For an electron, the magnetic moment is given by

$$\vec{\mu} = -\frac{e}{m}\vec{S}$$

(in SI units) where m is the mass, $\vec{S}$ is the spin operator, and $e = 1.6 \times 10^{-19}$ C> 0 is the fundamental unit of electronic charge. Therefore, we write the Hamiltonian as

$$H = \frac{e}{m}\vec{S} \cdot \vec{B} = \frac{eB}{m}S_z \equiv \omega S_z$$

where we have chosen $\vec{B}$ to be in the z direction. Therefore, for S_z eigenstates $|+\rangle$ and $|-\rangle$ we have $E_\pm = \pm\frac{1}{2}\hbar\omega$ and

$$|\alpha, t_0 = 0; t\rangle = c_+ \exp\left(-\frac{i\omega t}{2}\right)|+\rangle + c_- \exp\left(\frac{i\omega t}{2}\right)|-\rangle$$

which is Sakurai's equation (2.1.56), where the initial state is given by

$$|\alpha, t_0 = 0; t\rangle = c_+|+\rangle + c_-|-\rangle$$

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Now consider an initial state that is the spin "up" state of the S_x operator. That is

$$|\alpha, t_0 = 0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

(Sakurai equation (1.4.17a)). Then the probability of observing the spin in an S_x "up" or "down" state at a later time t is

$$|\langle S_x \pm |\alpha, t_0 = 0; t\rangle|^2 = \begin{array}{ll} \cos^2\frac{\omega t}{2} & \text{for} \quad |S_x+\rangle \\ \sin^2\frac{\omega t}{2} & \text{for} \quad |S_x-\rangle \end{array}$$

Note that the eigenstates of the S_x operator are given by Sakurai in equation (1.4.17a):

$$|S_x\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$$

The algebra is straightforward, and outlined at Sakurai near equation (2.1.60). The sum of these two results is unity, which it must be, since these are the only two possible measurements so the sum of their probability must be one. The results also show that the spin "rotates" around, coming back to unity for x-spin "up" when $t = 4\pi/\omega$.

Now, in order to calculate $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$, Sakurai cites equation (1.4.6), but be careful of the notation. The eigenstates to use for this expansion are those of the S_x operator, not $|+\rangle$ and $|-\rangle$. Therefore, we find

$$\begin{aligned}
\langle S_x\rangle_t &= \left(+\frac{\hbar}{2}\right)|\langle S_x + |\alpha, t_0 = 0; t\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle S_x - |\alpha, t_0 = 0; t\rangle|^2 \\
&= \left(\frac{\hbar}{2}\right)\cos^2\frac{\omega t}{2} - \left(\frac{\hbar}{2}\right)\sin^2\frac{\omega t}{2} = \left(\frac{\hbar}{2}\right)\cos\omega t
\end{aligned}$$

where we used the previous result for $|\langle S_x \pm |\alpha, t_0 = 0; t\rangle|^2$. For $|\langle S_y \pm |\alpha, t_0 = 0; t\rangle|^2$ and $\langle S_y\rangle$ refer to Sakurai equation (1.4.17b) and repeat the procedure to find

$$\langle S_y\rangle = \left(\frac{\hbar}{2}\right)\sin\omega t$$

Also, since $|\langle\pm|\alpha, t_0 = 0; t\rangle|^2 = \frac{1}{2}$ (which is very easy to show), we see that

$$\langle S_z\rangle = \left(\frac{\hbar}{2}\right)\frac{1}{2} - \left(\frac{\hbar}{2}\right)\frac{1}{2} = 0$$

This means that at $t = 0$, $\langle S_x\rangle = +\hbar/2$ (which was the initial condition) and then it rotates with the frequency ω in the xz plane.