PHYS6510 *Quantum Mechanics I*

Fall 2001 Problem Set #3

Due at Start of Class on *Tuesday* October 9

1. The ammonia molecule (NH_3) "vibrates" at a precise frequency, due to its molecular structure and the principles of quantum mechanics. Explain what this means, including looking up and making the connection between the vibration frequency and the eigenvalues of the Hamiltonian, and describe at least one historical use of this phenomenon. You may find it useful to refer to problems in this and previous homework assignments.

2. (Sakurai 2.9) A box containing a particle is divided into a right and left compartments by a thin partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket $|R\rangle$ ($|L\rangle$), where we have neglected spatial variations within each half of the box. The most general state vector can then be written as
$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle$$
where $\langle R|\alpha\rangle$ and $\langle L|\alpha\rangle$ can be regarded as "wave functions". The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian
$$H = \Delta\left(|L\rangle\langle R| + |R\rangle\langle L|\right)$$
where Δ is a real number with the dimension of energy.

   a. Find the normalized energy eigenkets. What are the corresponding energy eigenvalues?

   b. In the Schrödinger picture the base kets $|R\rangle$ and $|L\rangle$ are fixed, and the state vector moves with time. Suppose the system is represented by $|\alpha\rangle$ as given above at $t = 0$. Find the state vector $|\alpha, t_0 = 0; t\rangle$ for $t > 0$ by applying the appropriate time evolution operator to $|\alpha\rangle$.

   c. Suppose at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?

   d. Write down the coupled Schrödinger equations for the wave functions $\langle R|\alpha, t_0 = 0; t\rangle$ and $\langle L|\alpha, t_0 = 0; t\rangle$. Show that the solutions to the coupled Schrödinger equations are just what you expect from (b).

   e. Suppose the printer made an error and wrote H as
$$H = \Delta|L\rangle\langle R|$$
   By explicitly solving the most general time evolution problem with this Hamiltonian, show that probability conservation is violated. *Why? What's wrong?*

3. In class, we derived the position space representation of the $n = 0$ state of the simple harmonic oscillator, that is the wave function $u_0(x') \equiv \langle x'|0\rangle$. We then showed that one could directly derive the wave function $u_n(x') \equiv \langle x'|n\rangle$ from the wave function $u_{n-1}(x')$. Carry out this procedure for $n = 1, 2, 3$ and show that you get same result as by solving the Schrödinger wave equation, in terms of Hermite polynomials. (These solutions are given in Sakurai Sec.A.4 and in Merzbacher Sec.5.3.)

4. (From Sakurai 2.18) A coherent state of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a:

$$a|\lambda\rangle = \lambda|\lambda\rangle$$

where λ is, in general, a complex number.

   a. Prove that
   $$|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}|0\rangle$$
   is a normalized coherent state. *What are the allowed values of* λ*?*

   b. Prove that the minimum uncertainty relation is satisfied for such a state.

   c. Expand $|\lambda\rangle$ in terms of the simultaneous eigenstates of N and H:
   $$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle$$
   Show that the probability distribution $|f(n)|^2$ is in fact a Poisson distribution. Find the most probable value of n, hence of E.

   d. This formalism leads to minimum uncertainty states whose individual uncertainties in x and p are controllable, so called "squeezed states". Locate at least one published paper that discusses experimental observations of these states.

5. (Sakurai 2.21) A particle in one dimension is trapped between two rigid walls:

$$V(x) = \begin{cases} 0, & \text{for } \; 0 < x < L \\ \infty, & \text{for } \; x < 0,\; x > L \end{cases}$$

At $t = 0$ it is known to be exactly at $x = L/2$ with certainty. What are the *relative* probabilities for the particle to be found in various energy eigenstates? Write down the wave function for $t \geq 0$. (You need not worry about absolute normalization, convergence, and other mathematical subtleties.)

6. (Merzbacher Ex.4.17) Show that the substitution $\vec{\nabla} \to \vec{\nabla} - (iq/\hbar c)\vec{A}$ in [Merzbacher Eqn.](3.6) [the definition of the probability current density] produces a gauge-invariant current density and that this new $\vec{j}$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

for the Schrödinger equation [Merzbacher Eqn.](4.101) in the presence of an electromagnetic field.