

**Using Semidefinite Programming to  
Rebalance a Portfolio in the Presence of  
Transaction Costs**

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**Abstract**

In portfolio optimization, an acceptable balance between risk and reward is desired. Risk is represented by a quadratic objective function and reward by a linear constraint. Transaction costs can be modelled by imposing complementarity requirements on some variable pairs. We describe a semidefinite programming relaxation for the problem of rebalancing a portfolio with transaction costs and use this to develop a heuristic to find good portfolios. Computational results on real-world data sets are given.

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## 1 Introduction

- Seeking to invest in a **portfolio** of financial instruments: stocks, bonds, treasury bills, etc.
- Classic portfolio problem due to **Markowitz**: Want to minimize risk subject to achieving a certain expected return.
- Solving this problem for different desired expected returns gives the **efficient frontier**.
- Note: We do not consider short selling in our model.

**Markowitz** proposed solving a **quadratic programming problem** in order to determine an appropriate asset allocation:

$$\begin{array}{ll} \min & \frac{1}{2}x^T Q x \\ \text{subject to} & \mu^T x \geq \alpha \\ & e^T x = 1 \\ & x \geq 0 \end{array}$$

where

- $x_i$  is the **proportion of assets** that are placed in financial instrument  $i$ .
- $\mu$  is the **expected return** of the financial instruments.
- $Q$  is the **covariance matrix** of the financial instruments.
- $\alpha$  is the **desired return**.
- $e$  is the vector of all ones.

## 2 Transaction costs

- Typically, we already have a collection of assets.
- There are **transaction costs** associated with changing the proportion of assets in any asset class.
- Transaction costs could be
  - **Proportional:** The cost is a certain percentage of the value of each transaction.
  - **Fixed:** There is a single cost for a transaction, regardless of the size of the transaction.
- Costs can differ for **buying** and **selling**.
- Costs can measure more than just brokerage costs. For example, they can
  - measure our **confidence of our estimates** of the expected return of the stock.
  - measure the **tax implications** of selling part of a particular holding.

### **3 Why Rebalance?**

If we previously assigned our assets optimally, why would we have to change the proportions?

- Over time, the proportions will change. For example, the proportion of funds in the asset with the largest return will increase.
- Updated information about the risks and returns of the financial instruments may become available.
- The set of possible investment choices may change.

## 4 Modelling choices

### 4.1 Limit the number of transactions

- Limiting the number of transactions limits the transaction costs paid, obviously!
- This approach is especially applicable if the transaction costs are **fixed**, so there is a single cost for a transaction regardless of the size of the transaction.
- Can be modeled as an **integer programming problem** with a quadratic objective function. Need to introduce binary variables for each asset indicating whether there is a transaction in that asset.

## 4.2 Place the costs in the objective

- It is possible to find the whole efficient frontier by constraining the risk we are prepared to accept and then maximizing the expected return.
- This gives a quadratically constrained quadratic program, which can be solved efficiently using an **interior point method**.
- Transaction costs can be **subtracted from returns** in the objective function.
- **Advantage**: easy to solve.
- Principal **drawback**: forces a particular period for **amortization** of costs.
- This approach is used by **Axioma**.

### 4.3 Our method

- We incorporate the transaction costs into the constraint  $e^T x = 1$ .
- Formally, this is done by introducing a dummy variable  $x_0$ , representing the amount paid in transaction costs. This **dummy asset** has an expected return of zero, a risk of zero, and covariance of zero with any other security.

## 5 Modelling details

- The **current proportions** in each security are represented by  $\bar{x}$ . Note that  $e^T \bar{x} = 1$ .
- Let  $x$  be the desired proportion in each asset.
- We define  $u$  and  $v$  so that  $x - \bar{x} = u - v$ , with  $u$  and  $v$  nonnegative.
- Let  $c_i^B$  and  $c_i^S$  be the proportional transaction costs for buying and selling asset  $i$ , respectively. **Total transaction costs** are  $\sum_i c_i^B u_i + \sum_i c_i^S v_i$ .

After eliminating the dummy variable  $x_0 = (c^B)^T u + (c^S)^T v$ , get a quadratic program:

$$\begin{aligned} \min_{u,v} \quad & \frac{1}{2}(\bar{x} + u - v)^T Q(\bar{x} + u - v) \\ \text{subject to} \quad & \mu^T(\bar{x} + u - v) \geq \alpha \\ & (c^B + e)^T u + (c^S - e)^T v = 0 \\ & 0 \leq u \leq e - \bar{x} \\ & 0 \leq v \leq \bar{x} \quad . \end{aligned}$$

## The transaction cost constraint

$$(c^B + e)^T u + (c^S - e)^T v = 0$$

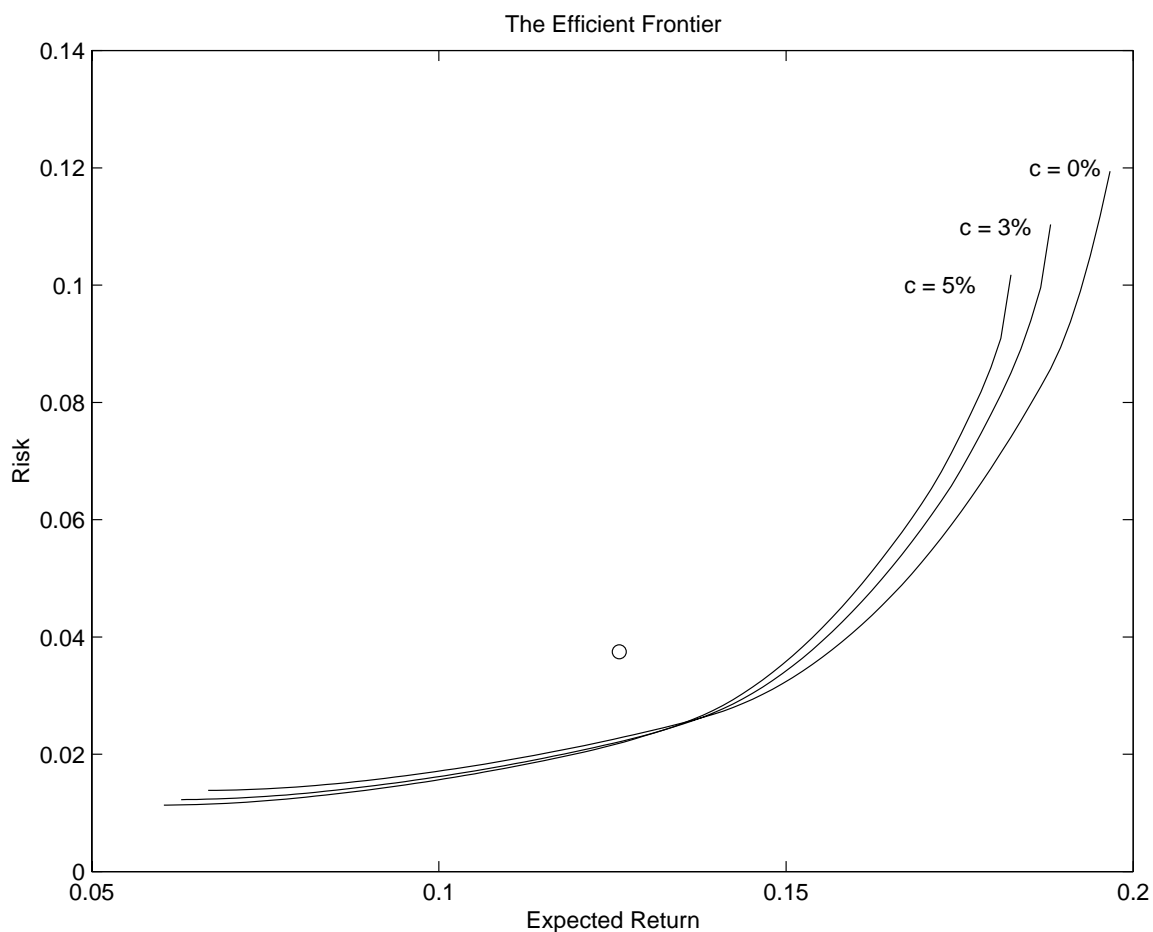
- The total assets we have available,  $e^T(\bar{x} + u - v)$ , are decreased by the transaction costs.
- In effect, when a quantity  $v_i$  is sold, the amount available to purchase other securities is  $(1 - c_i^S)v_i$ .
- Similarly, when a quantity  $u_i$  of asset  $i$  is purchased, the total available to purchase other securities is decreased by  $(1 + c_i^B)u_i$ .

## 6 Crossover

The crossover is the result of a modelling error. We needed to make two modifications:

1. Enforce complementarity: it is not allowed to **both buy and sell** a particular security. It was sometimes advantageous to do this, because the risk was reduced.
2. Rescale the objective to reflect **risk per dollar invested**.

Figure 1:  
**Multiple “Efficient” Frontiers using Same Initial Portfolio**



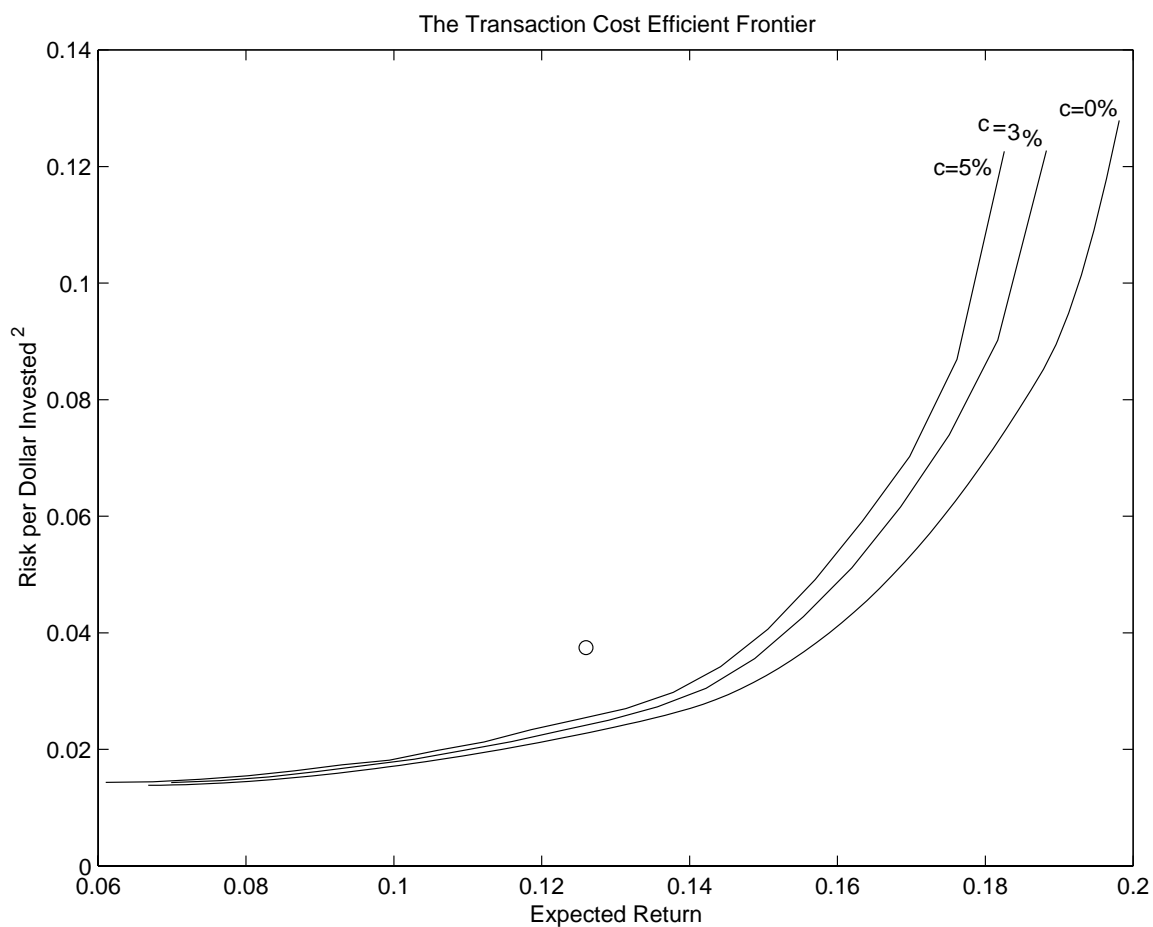
- The initial portfolio is located by a circle.
- Notice that as the level of transaction costs  $c$  increases, the frontier shifts left corresponding to a decrease in return for a given level of risk.
- Also, notice that the efficient frontiers for different transaction costs cross.

Get new model:

$$\begin{aligned}
 \min_{u,v} \quad & \frac{(\bar{x}+u-v)^T Q (\bar{x}+u-v)}{2(1-(c^B)^T u - (c^S)^T v)^2} \\
 \text{subject to} \quad & \mu^T (\bar{x} + u - v) \geq \alpha \\
 & (c^B + e)^T u + (c^S - e)^T v = 0 \\
 & u^T v = 0 \\
 & 0 \leq u \leq e - \bar{x} \\
 & 0 \leq v \leq \bar{x} \quad .
 \end{aligned}$$

A technique of Charnes and Cooper can be used to replace the fractional objective with a quadratic objective. So we get a **quadratic programming problem with a complementarity constraint**.

Figure 2:  
**Multiple Efficient Frontiers using Same Initial Portfolio**



- The initial portfolio is again located by a circle.
- Notice that as the level of transaction costs  $c$  increases, the curves shift left.
- However, crossover no longer occurs.
- Increased transaction costs reduce investment choice.

## 7 Solving Quadratic Programs with a Complementarity Constraint

(Warning: different notation ahead.)

Our quadratic program can be cast in the form:

$$\begin{aligned}
 \min_{x,y} \quad & \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} Q & d \\ d^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
 \text{subject to} \quad & Ax - by = 0 \\
 & y = 1 \qquad \qquad \qquad (QPC) \\
 & x_i x_j = 0 \quad \forall (i, j) \in \Phi \\
 & 0 \leq x \leq u
 \end{aligned}$$

where the set  $\Phi$  holds the complementarity relationships.

Here,  $x$  is a vector and  $y$  is a homogenizing scalar variable.

If we knew which of  $x_i$  and  $x_j$  was zero in the optimal solution to  $(QPC)$  for all  $(i, j) \in \Phi$  then we could solve  $(QPC)$  by solving a **quadratic programming problem**. Could examine all the candidate subproblems, but this is too expensive.

## 8 Semidefinite programming relaxation

Let  $X := xx^T$ , so  $X_{ij} = x_i x_j$ . Define

$$Z := \begin{pmatrix} X & yx \\ yx^T & y^2 \end{pmatrix}.$$

Define

$$C := \begin{pmatrix} Q & d \\ d^T & 0 \end{pmatrix}.$$

Then the **objective function** of (QPC) is equal to  $C \bullet Z$ , where  $\bullet$  denotes the Frobenius inner product.

The **complementarity constraints** are equivalent to  $Z_{ij} = 0$  for  $(i, j) \in \Phi$ . This is the **bullet operator** of Wolkowicz *et al.*

**Bounds:** Require  $X_{ii} \leq u_i^2$  for each  $i$ . Require all entries of  $Z$  to be nonnegative.

Require  $Z$  to be **positive semidefinite**. This forces  $X_{ij} \leq u_i u_j$ .

Require  $y^2 = 1$ .

**Lifting** the constraint  $Ax - by = 0$ :

- Take the outer product of both sides of this constraint with the vector  $\begin{pmatrix} x \\ y \end{pmatrix}$ .
- We get  $\binom{n+1}{2}$  homogeneous linear constraints on the entries of  $Z$ .
- These constraints can be written

$$(a_j e_l^T + e_l a_j^T) \bullet Z = 0$$

where  $a_j$  is the  $j$ th row of  $[A \quad -b]$ , written as a column vector, and  $e_l$  is the  $l$ th unit vector.

- From  $l = n + 1$ , we get the constraints  $Ax - by = 0$ .

Get the **semidefinite programming relaxation**:

$$\begin{aligned} \min \quad & C \bullet Z \\ \text{subject to} \quad & A_k \bullet Z = b_k \text{ for } k = 1, \dots, K \quad (SDP) \\ & Z \succeq 0 \end{aligned}$$

where  $K$  is the number of linear constraints on  $Z$ .

- If the solution matrix  $Z$  is **rank one**, then it leads directly to a solution to  $(QPC)$ .
- **Algorithm:** Solve this semidefinite relaxation of  $(QPC)$ . Use the solution to  $(QPC)$  to generate candidate quadratic programming subproblems. Solve the quadratic programs and report the best solution found.

## 9 Selecting quadratic programming subproblems

- We want to decide (heuristically) whether we can **fix any of the variables** in the complementarity relationships. This will let us reduce the number of candidate subproblems.
- Let  $Z^*$  be the optimal solution to (SDP). (Note: we can instead use an approximate solution to (SDP) for large problems.)
- **Spectral decomposition** of  $Z^*$ :

$$Z^* = \sum_i \lambda_i \xi_i \xi_i^T$$

where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n+1} \geq 0$  are the eigenvalues of  $Z^*$ .

- The closest **rank-one** matrix to  $Z^*$  is  $Z^1 := \lambda_1 \xi_1 \xi_1^T$ .
- Choose  $x^* := \sqrt{\lambda_1} \xi_1$  as our estimate of the solution to (QPC).
- Let  $x_i$  and  $x_j$  be a complementary pair of variables. If  $x_i^*$  is close to zero and  $x_j^*$  is close to its upper bound then **only consider subproblems where  $x_i$  is forced to be zero**.

## 10 Results on a nine security problem

Transaction costs are 5%. Results are confirmed by complete enumeration of all subproblems.

Required Expected Return (%)	(rSDP) Objective Value	Minimal QP-subproblem Objective Value	Number of QP-subproblems Considered
18.26	0.043725	0.122595	solved as LP
17.62	0.037793	0.086903	16
16.98	0.031234	0.070248	16
16.34	0.025977	0.058898	16
15.70	0.021162	0.049131	16
15.06	0.018161	0.040815	16
14.42	0.015173	0.034284	16
13.78	0.013183	0.029781	16
13.14	0.011649	0.027103	16
12.50	0.010877	0.024602	16
11.86	0.010024	0.023381	32
11.22	0.009383	0.021267	32
10.58	0.008607	0.019820	32
9.94	0.007897	0.018517	32
9.30	0.007363	0.017363	32
8.66	0.006915	0.016326	32
8.02	0.006528	0.015471	16
7.38	0.006182	0.015153	16
6.74	0.005994	0.014462	16
6.68	0.005937	0.014426	16
6.10	0.005909	0.013850	8

Table 1: **Performance for the Markowitz  $c=5\%$  TCEF:** Since 9 securities were involved, the fourth column compares against a total of  $2^9 = 512$  possible QP-subproblems. For most cases, correct decisions were made for 5 out of the 9 securities. The optimal solution was found in every case by the algorithm.

## Comparison of the quadratic subproblems generated when required expected return is 12.5%

Optimal Objective Values for the 10 Best QP-subproblems	Optimal Objective Values for QP-subproblems Selected by the Strategy	Comment on Relationship
0.024602	0.024602	Strategy selected true optimal solution.
0.024703		Not selected, see Table 3.
0.024722	0.024722	Selected by our strategy.
0.024737	0.024737	Selected by our strategy.
0.024875	0.024875	Selected by our strategy.
0.024877	0.024877	Selected by our strategy.
0.025168	0.025168	Selected by our strategy.
0.025186		
0.025238		
0.025240		
	0.033665	Feasible but not close to optimal .
	0.034617	Feasible but not close to optimal .
	0.035161	Feasible but not close to optimal .

Table 2: **Distribution of QP-subproblems Selected by the Strategy**

Notice that the QP-subproblems selected by our strategy are densely clustered near the true optimum. More specifically, the strategy selected the true optimal and 6 other subproblems located within 2% of the true optimum.

## Comparing the two best QP subproblems

Objective Values for the 2 Best QP-subproblems	Security Number								
	1	2	3	4	5	6	7	8	9
0.024602	Sell	Sell	Buy	Sell	Buy	Sell	Buy	Sell	Sell
0.024703	Sell	Buy	Buy	Sell	Buy	Sell	Buy	Sell	Sell

Table 3: Comparison of Active Set Choices for QP-subproblems

- This table contains the active set choices for the **two best QP-subproblems**.
- All decisions are the same except for security #2.
- The second best problem was constrained to **buy security #2**. The optimal solution to this subproblem required **buying zero** of this security.
- Thus, the optimal solution to the second best subproblem is feasible in the best subproblem.

## 11 Handling degenerate decisions

- Given a pair  $(i, j) \in \Phi$ , a **degenerate case** is when we impose the constraint  $x_i = 0$  in the quadratic programming subproblem, and the solution to the subproblem has  $x_j = 0$ .
- Occasionally, the algorithm does not choose the **optimal** quadratic subproblem. In all our test cases, this was indicated by one of the chosen subproblems having a degenerate solution.
- **Undoing** the decision to fix  $x_i = 0$  was enough to enable the algorithm to find the optimal solution.
- More aggressive choices are possible: only examine selected subproblems with  $x_i$  allowed to be nonzero. Thus, we would only look at **neighbours** of the problem that exhibited degeneracy.

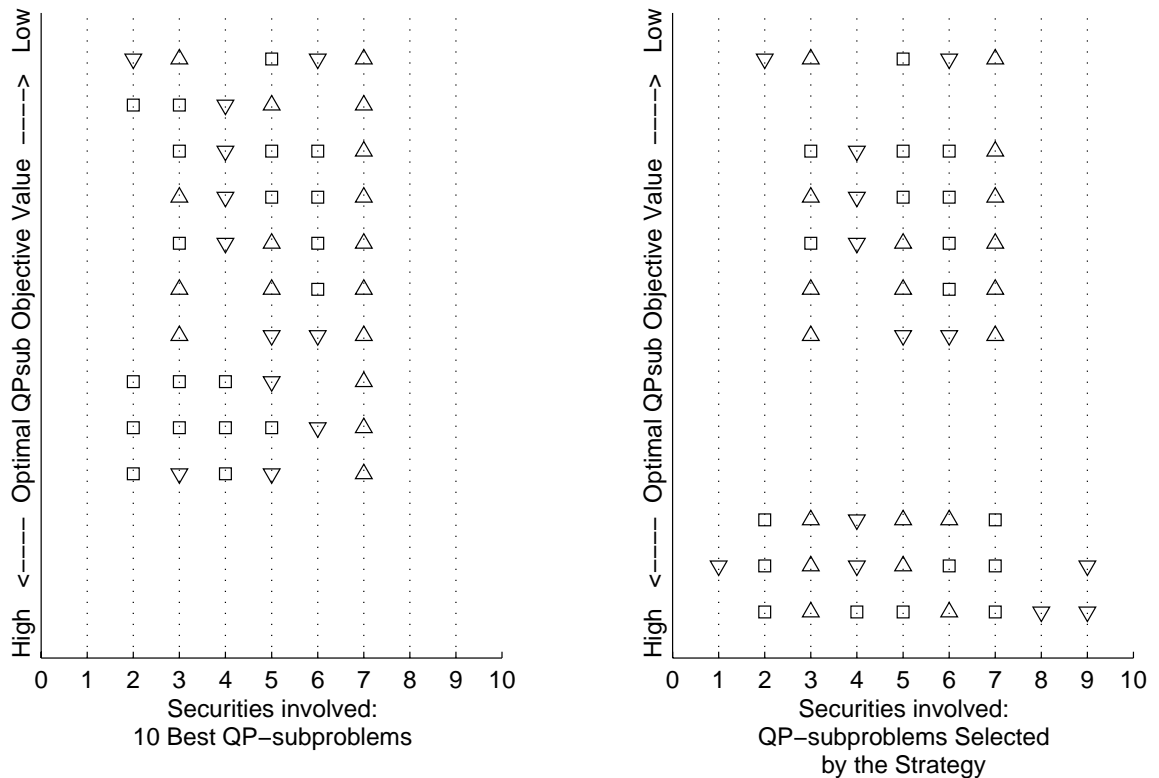
## **12 The Dow Jones**

- Data collected for eight years up through the end of 2000. The returns were exceptional in this period.
- Used the thirty stocks in the Dow Jones Industrial Average at the end of the period.
- Overlapping annual returns were calculated to increase the number of datapoints used to construct the covariance matrix.
- Initial allocation was randomly generated.
- Buying and selling costs for individual securities were randomly generated, between 0 and 5.

## 13 Conclusions

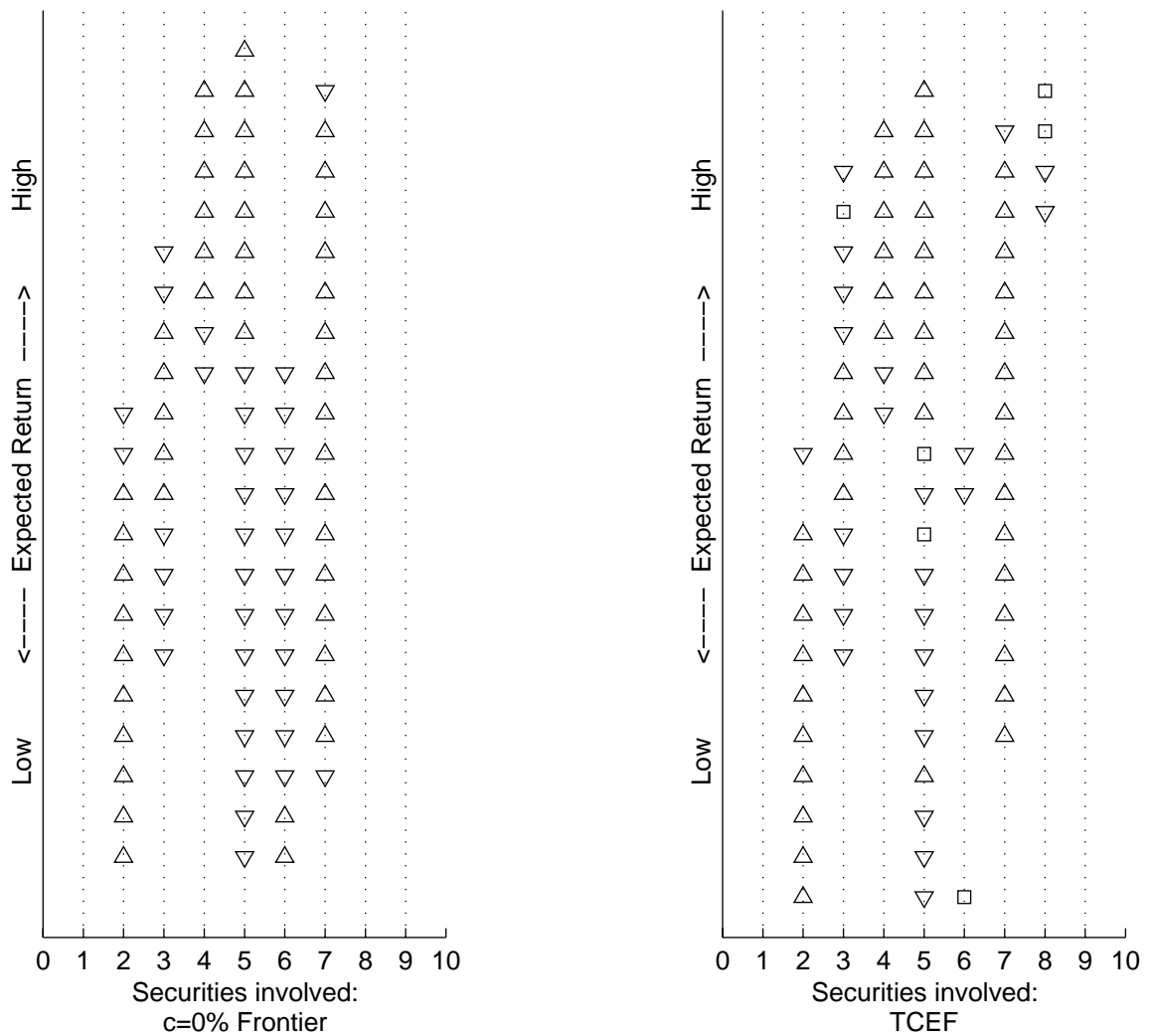
- We've shown that solving a semidefinite programming problem can provide useful heuristic information for solving quadratic programming problems with complementarity constraints.
- This can be applied to portfolio optimization problems with transaction costs.
- Our formulation of the portfolio problem is an example of a mathematical program with equilibrium constraints (MPEC), and our solution method can be used for more general types of MPECs.

Figure 3:  
**Comparison of QP-subproblem Optimal Solutions**



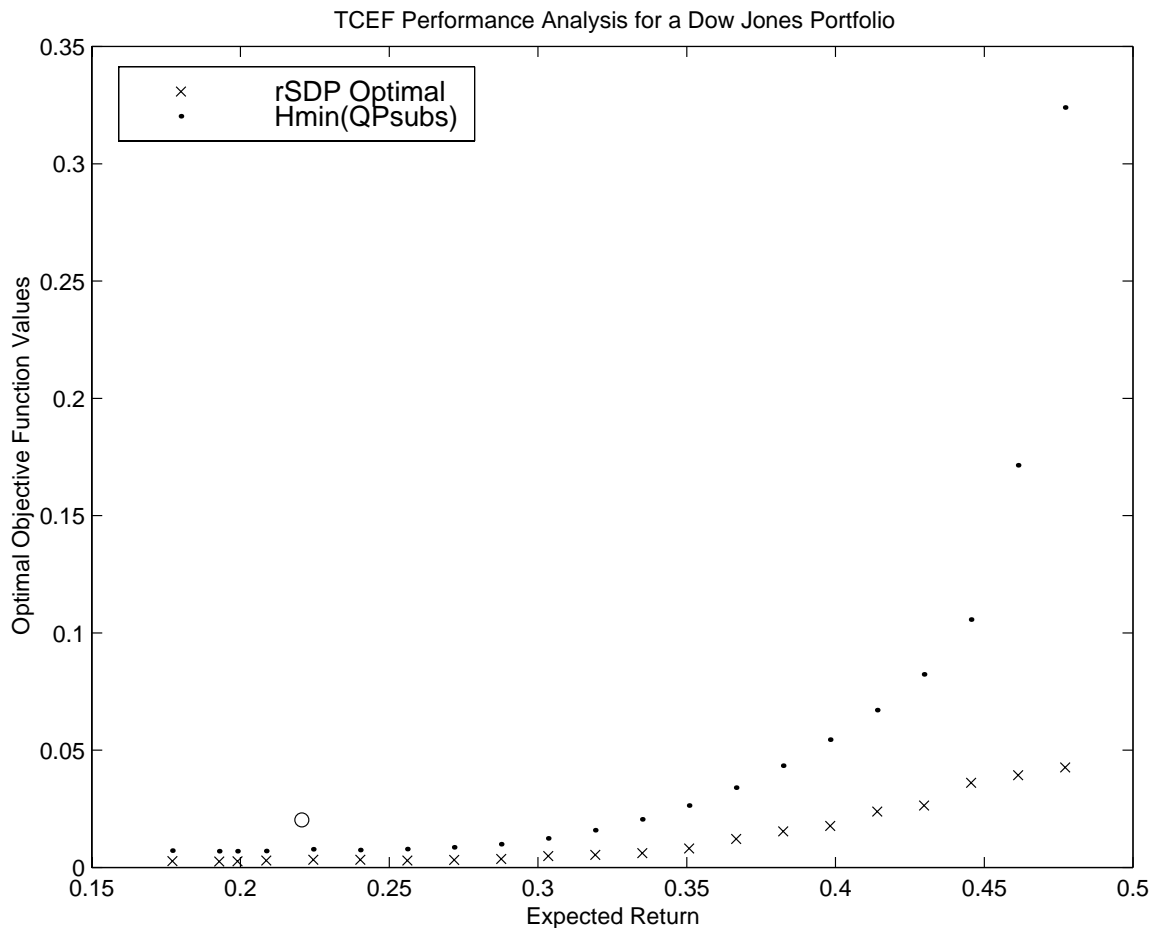
- Securities numbered 1 through 9 exist.  
 No decisions made for securities #3, 5, 6, and 7.
- The **optimal solution to the second subproblem is feasible in the first subproblem.**
- Notation: bought  $\triangle$ , sold  $\nabla$ , unchanged  $\square$ .  
 No symbol: entire allocation liquidated.

## Visualization of the involved securities



- These panels show which securities are involved along the no-transaction cost efficient frontier (left) and the additional TCEF (right).
- Notation: bought  $\triangle$ , sold  $\nabla$ , unchanged  $\square$ .
- The absence of a symbol means that the entire allocation of that particular security was liquidated.

# Computational Results for the Dow Jones Portfolio



- The value of the SDP relaxation and of the best QP subproblem examined are given for each return requirement.
- The circle again locates the initial portfolio.
- Notice that the (rSDP) optimal objective value does not mirror the shape of the TCEF, particularly for higher required expected returns.
- However, the TCEF itself has no visible defects. This provides strong empirical support for our strategy.