1. Section 3.6, Page 225, question 22.

For A to be nonsingular, the column vectors in A must be a linearly independent set :

Given : $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is an orthogonal set, is the only solution to* $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ $c_1 = c_2 = c_3 = 0$?

$$\mathbf{u}_1^T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) = \mathbf{u}_1^T\mathbf{0}$$

$$\mathbf{u}_1^T c_1\mathbf{u}_1 + \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Since $\mathbf{u}_1$ *is not a zero vector*, $\mathbf{u}_1^T\mathbf{u}_1 = \|\mathbf{u}_1\|^2 \neq 0$, *and* $c_1 = 0$

Similarly, $c_2 = c_3 = 0$

Hence, A is nonsingular.

$$A^T A = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$$

Since the vectors are orthogonal, by definition :

$$A^T A = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & 0 & 0 \\ 0 & \mathbf{u}_2^T\mathbf{u}_2 & 0 \\ 0 & 0 & \mathbf{u}_3^T\mathbf{u}_3 \end{bmatrix} \text{ which is a diagonal matrix.}$$

$$A^T A(\text{Exercise } 1) = \begin{bmatrix} 1^2 + (1)^2 + (1)^2 & 0 & 0 \\ 0 & (-1)^2 + 0^2 + 1^2 & 0 \\ 0 & 0 & (-1)^2 + 2^2 + (-1)^2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

2. Section 3.7, Page 240, question 18.

$$\text{Given}: W = \{\mathbf{x} \text{ in } R^3 : \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_2 = x_3 = 0\} \therefore W = \{\mathbf{x} \text{ in } R^3 : \mathbf{x} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = x_3 = 0\} \therefore \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is a basis for } W.$$

$$\text{Orthonormal basis for } W : \mathbf{u_1} = \mathbf{w_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is both an orthogonal and an orthonormal basis for } W.$$

$$T(\mathbf{v}) = (\mathbf{v}^T\mathbf{u_1})\mathbf{u_1} = \begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

v is any vector in 3 dim*ensional space. W is the x* − *axis.* $T(v)$ *is the projection of v on the x* − *axis.*

3. Section 3.7, Page 242, question 34. Show also that T is orthogonal.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

Linear Transformation because:

$$T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}\right) = \begin{bmatrix} a_2+b_2 \\ a_1+b_1 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} = T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) \therefore T(\mathbf{a}+\mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b})$$

$$T\left(a\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} au_1 \\ au_2 \end{bmatrix}\right) = \begin{bmatrix} au_2 \\ au_1 \end{bmatrix} = a\begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = aT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) \therefore T(a\mathbf{u}) = aT(\mathbf{u})$$

$\therefore T$ *is a linear transformation.*

$$\left\| T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \right\| = \sqrt{{x_1}^2 + {x_2}^2} = \left\| \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \right\| \therefore T \text{ is orthogonal.}$$

4. Section 4.1, Page 279, question 4.

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix}$$

To be singular: $(1-\lambda)(4-\lambda)+2=0 \Rightarrow \lambda^2 - 5\lambda + 4 + 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3)=0$

$\therefore \boxed{\lambda = 2, \lambda = 3}$

Eigenvectors:

Let: $\lambda = 2$, *Solve* $(A - \lambda I)\mathbf{x} = 0$

$$\begin{bmatrix} -1 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \underset{R_2+R_1}{\Rightarrow} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow -x_1 = 2x_2 \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \boxed{x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ for arbitrary } x_2 \neq 0}$$

Let: $\lambda = 3$, *Solve* $(A - \lambda I)\mathbf{x} = 0$

$$\begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \underset{R_2+\frac{1}{2}R_1}{\Rightarrow} \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow -2x_1 = 2x_2 \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = \boxed{x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ for arbitrary } x_2 \neq 0}$$

5. (MAPLE) Bauldry *et al.*, page 161, question 2.

```
> with(LinearAlgebra):
> GramSchmidt([<1,0,2,3,1>,<1,0,2,1,3>,<1,2,3,1,0>], normalized);
```

$$\left[\begin{bmatrix} \frac{1}{15}\sqrt{15} \\ 0 \\ \frac{2}{15}\sqrt{15} \\ \frac{1}{5}\sqrt{15} \\ \frac{1}{15}\sqrt{15} \end{bmatrix}, \begin{bmatrix} \frac{1}{195}\sqrt{390} \\ 0 \\ \frac{2}{195}\sqrt{390} \\ -\frac{3}{130}\sqrt{390} \\ \frac{17}{390}\sqrt{390} \end{bmatrix}, \begin{bmatrix} \frac{4}{2795}\sqrt{5590} \\ \frac{2}{215}\sqrt{5590} \\ \frac{21}{2795}\sqrt{5590} \\ -\frac{23}{5590}\sqrt{5590} \\ -\frac{23}{5590}\sqrt{5590} \end{bmatrix} \right]$$

6. (MAPLE) Bauldry *et al.*, page 73, question 6.

```
> V := Matrix([[2,3,5,2],[1,2,1,3],[2,4,1,5],[3,1,2,3]]);
```

$$V := \begin{bmatrix} 2 & 3 & 5 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 5 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

```
> Determinant(V);
```

$$-30$$

```
> V2:= Matrix([[2,3,5,2.01],[1,2,1,3],[2,4,1,5],[3,1,2,3]]);
```

$$V2 := \begin{bmatrix} 2 & 3 & 5 & 2.01 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 5 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

```
> Determinant(V2);
```

$$-29.95$$

The determinant changed by .05 or five times the change in the one entry. Overall, the changes that occur to the determinant are rather small in magnitude (one change by .01 in one instance caused the determinant to become -29.58) and can either add or subtract from the determinant.

```
> N:=Matrix([[1,2,3],[4,5,6],[7,8,9]]);
```

$$N := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

```
>  Determinant(N);
```

$$0$$

```
> N:=Matrix([[1,2,3.01],[4,5,6],[7,8,9]]);
```

$$N := \begin{bmatrix} 1 & 2 & 3.01 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

```
>  Determinant(N);
```

$$-0.03$$

The changes are again of small magnitude and either add to or subtract from the determinant.

You don't expect it's determinant to be zero, as this is a very rare/special case in terms of probabilities. Therefore, you expect that randmatrix will most often generate matrices that are invertible.