1. Section 1.9, Page 103, question 48.

To make the matrix nonsingular, $[\mathrm{A} \mid \boldsymbol{\theta}]$ must lead to a system, $\mathrm{A}\mathbf{x} = \boldsymbol{\theta}$ whose only solution is the trivial solution - namely: $\mathbf{x} = \boldsymbol{\theta}$.

$$[\mathrm{A} \mid \boldsymbol{\theta}] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & a & 0 \end{bmatrix} \underset{R_3 - R_1}{\Rightarrow} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & a+1 & 0 \end{bmatrix} \overset{R_1 - R_2}{\Rightarrow} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & a+1 & 0 \end{bmatrix}$$

Since the last matrix shown is row equivalent to $[\mathrm{A} \mid \boldsymbol{\theta}]$, by theorem 18 we know that for A to be nonsingular, A has to be row equivalent to I, the Identity matrix. This can only happen if the quantity $a + 1 \neq 1$.

Therefore, for A to be nonsingular, $\boxed{a \neq -1}$.

2. Section 3.2, Page 175, question 30.

Given that U and V are subspaces in R^n, for U+V to be a subspace in R^n the following conditions must be met:

1. U+V contains the **0** vector.

   Since U is a subspace it contains the zero vector. Since V is a subspace, it contains the **0** vector. Hence **0**+**0** = **0** shows that the zero vector exists in U+V. Thus **condition 1 is met**.

2. Given vector **x** and vector **y** that are in U+V, x+y must also be in U+V.

   $\mathbf{x} = \mathbf{u}_1 + \mathbf{v}_1 \quad \mathbf{y} = \mathbf{u}_2 + \mathbf{v}_2$

   $(\mathbf{x} + \mathbf{y}) = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) \;\therefore$ *Condition Satisfied* since the u and v vectors are in a subspace and hence closed for scalar addition.

3. Given vector **x** is in U+V, for any scalar a, a**x** must be in U+V.

   $\mathbf{x} = \mathbf{u} + \mathbf{v}$

   $a(\mathbf{x}) = a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \therefore$ *Condition Satisfied* since the u and v vectors are in a subspace and hence closed for scalar multiplication.

3. Section 3.3, Page 187, question 34.

$$\begin{bmatrix} 1 & -2 & 1 & b_1 \\ 2 & -3 & 5 & b_2 \\ 1 & 0 & 7 & b_3 \end{bmatrix} \Rightarrow \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 2 & 6 & b_3 - b_1 \end{bmatrix} \begin{matrix} R_1 + 2R_2 \\ \Rightarrow \\ R_3 - 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 7 & b_1 + 2(b_2 - 2b_1) \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - 2(b_2 - 2b_1) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 7 & 2b_2 - 3b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & 3b_1 - 2b_2 + b_3 \end{bmatrix}$$

Setting $\mathbf{b} = \mathbf{0}$ *yields* $x_1 = -7x_3$ *and* $x_2 = -3x_3$ *for arbitrary values of* $x_3 \therefore$

$$\boxed{N(A) = \{\mathbf{x} \text{ in } R^3 : x_1 + 7x_3 = 0 \text{ and } x_2 + 3x_3 = 0\}}$$

For $A\mathbf{x} = \mathbf{b}$ *to be consistent*, $3b_1 - 2b_2 + b_3 = 0 \therefore$

$$\boxed{R(A) = \{\mathbf{b} \text{ in } R^3 : 3b_1 - 2b_2 + b_3 = 0\}}$$

4. Section 3.3, Page 188, question 40.

$$\begin{bmatrix} 1 & 0 & 7 & 2b_2 - 3b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & 3b_1 - 2b_2 + b_3 \end{bmatrix}$$

$R(A) = \{\mathbf{b} \text{ in } R^3 : 3b_1 - 2b_2 + b_3 = 0\}$

$(i) 3b_1 - 2b_2 + b_3 = 3(1) - 2(2) + 0 = -1.$ *Therefore* $\boxed{\text{this } \mathbf{b} \text{ is not in } R(A).}$

$(ii) 3b_1 - 2b_2 + b_3 = 3(1) - 2(1) + (-1) = 0.$ *Therefore* $\boxed{\text{this } \mathbf{b} \text{ is in } R(A).}$

$$(b) \begin{bmatrix} 1 & 0 & 7 & 2(1)-3(1) \\ 0 & 1 & 3 & (1)-2(1) \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 7 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \boxed{\begin{bmatrix} -7x_3 - 1 \\ -3x_3 - 1 \\ x_3 \end{bmatrix}}$$

$$(c) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} + (0)\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \Rightarrow \mathbf{b} = \boxed{-A_1 - A_2}$$

$(iii) 3b_1 - 2b_2 + b_3 = 3(4) - 2(7) + 2 = 0.$ *Therefore* $\boxed{\text{this } \mathbf{b} \text{ is in } R(A).}$

$$(b) \begin{bmatrix} 1 & 0 & 7 & 2(7)-3(4) \\ 0 & 1 & 3 & (7)-2(4) \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \boxed{\begin{bmatrix} 2 - 7x_3 \\ -3x_3 - 1 \\ x_3 \end{bmatrix}}$$

$$(c) \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} + (0)\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \Rightarrow \mathbf{b} = \boxed{2A_1 - A_2}$$

$(iv) 3b_1 - 2b_2 + b_3 = 3(0) - 2(1) + 2 = 0.$ *Therefore* $\boxed{\text{this } \mathbf{b} \text{ is in } R(A).}$

$$(b) \begin{bmatrix} 1 & 0 & 7 & 2(1)-3(0) \\ 0 & 1 & 3 & (1)-2(0) \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \boxed{\begin{bmatrix} 2 - 7x_3 \\ 1 - 3x_3 \\ x_3 \end{bmatrix}}$$

$$(c) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} + (0)\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \Rightarrow \mathbf{b} = \boxed{2A_1 + A_2}$$

$(v) 3b_1 - 2b_2 + b_3 = 3(0) - 2(1) + (-2) = -4.$ *Therefore* $\boxed{\text{this } \mathbf{b} \text{ is not in } R(A).}$

$(vi) 3b_1 - 2b_2 + b_3 = 3(0) - 2(0) + (0) = 0.$ *Therefore* $\boxed{\text{this } \mathbf{b} \text{ is in } R(A).}$

$$(b) \begin{bmatrix} 1 & 0 & 7 & 2(0)-3(0) \\ 0 & 1 & 3 & (0)-2(0) \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \boxed{\begin{bmatrix} -7x_3 \\ -3x_3 \\ x_3 \end{bmatrix}}$$

$$(c) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (0)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (0)\begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} + (0)\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \Rightarrow \mathbf{b} = \boxed{0A_1 + 0A_2 + 0A_3}$$

5. Section 3.3, Page 188, question 48.

$Let: A=[S_1 \quad S_2 \quad S_3 \quad S_4]$

$$A=\begin{bmatrix}1&-2&1&-2\\0&0&1&3\\1&-2&2&1\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}1&0&1\\-2&0&-2\\1&1&2\\-2&3&1\end{bmatrix}\begin{matrix}\Rightarrow\\R_2+2R_1\\R_3-R_1\\R_4+2R_1\end{matrix}\begin{bmatrix}1&0&1\\0&0&0\\0&1&1\\0&3&3\end{bmatrix}\begin{matrix}\Rightarrow\\ \\ \\R_4-3R_3\end{matrix}\begin{bmatrix}1&0&1\\0&0&0\\0&1&1\\0&0&0\end{bmatrix}$$

Looking at the nonzero rows, the set $T=\{\mathbf{w_1},\mathbf{w_2}\}$ *such that* $Sp(S)=Sp(T)$ *is* $\boxed{\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}}$

6. Section 3.4, Page 201, question 22.

Solve dependence relation of $: x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\boldsymbol{\theta}$

$$[S\,|\,\boldsymbol{\theta}]=\begin{bmatrix}1&2&3&0\\2&1&2&0\end{bmatrix}\begin{matrix}\Rightarrow\\R_2-2R_1\end{matrix}\begin{bmatrix}1&2&3&0\\0&-3&-4&0\end{bmatrix}\begin{matrix}\Rightarrow\\-\frac{1}{3}R_2\end{matrix}\begin{bmatrix}1&2&3&0\\0&1&\frac{4}{3}&0\end{bmatrix}\begin{matrix}R_1-2R_2\\\Rightarrow\end{matrix}\begin{bmatrix}1&0&\frac{1}{3}&0\\0&1&\frac{4}{3}&0\end{bmatrix}\Rightarrow\begin{matrix}x_1=&\frac{1}{3}x_3\\x_2=&\frac{4}{3}x_3\end{matrix}$$

Since x_3 *is unconstrained, we can delete the* $\mathbf{v}_3$ *vector, leaving*

$$\begin{bmatrix}1&2&0\\2&1&0\end{bmatrix}\begin{matrix}\Rightarrow\\R_2-2R_1\end{matrix}\begin{bmatrix}1&2&0\\0&-1&0\end{bmatrix}\begin{matrix}\Rightarrow\\-R_2\end{matrix}\begin{bmatrix}1&2&0\\0&1&0\end{bmatrix}\begin{matrix}R_1-2R_2\\\Rightarrow\end{matrix}\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix},$$ *which clearly shows that*

$\boxed{\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}}$ *is a linear independent set that is a basis for* $Sp(s)$.

$$S^T=\begin{bmatrix}1&2\\2&1\\3&2\end{bmatrix}\begin{matrix}\Rightarrow\\R_2-2R_1\\R_3-3R_1\end{matrix}\begin{bmatrix}1&2\\0&-3\\0&-4\end{bmatrix}\begin{matrix}\Rightarrow\\-\frac{1}{3}R_2\\ \end{matrix}\begin{bmatrix}1&2\\0&1\\0&-4\end{bmatrix}\begin{matrix}R_1-2R_2\\\Rightarrow\\R_3+4R_2\end{matrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

$S=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\therefore \boxed{\{\mathbf{u_1},\mathbf{u_2}\}=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}}$ *is a set that is a basis for* $Sp(s)$.

7. Section 3.4, Page 201, question 32.

$$\begin{bmatrix}1&1&2&a\\-1&1&-3&b\\-2&2&-3&c\end{bmatrix}\begin{matrix}\Rightarrow\\R_2+R_1\\R_3+2R_1\end{matrix}\begin{bmatrix}1&1&2&a\\0&2&-1&a+b\\0&4&1&2a+c\end{bmatrix}\begin{matrix}\Rightarrow\\\frac{1}{2}R_2\\ \end{matrix}\begin{bmatrix}1&1&2&a\\0&1&-\frac{1}{2}&\frac{1}{2}(a+b)\\0&4&1&2a+c\end{bmatrix}\begin{matrix}R_1-R_2\\\Rightarrow\\R_3-4R_2\end{matrix}\begin{bmatrix}1&0&\frac{5}{2}&a-\frac{1}{2}(a+b)\\0&1&-\frac{1}{2}&\frac{1}{2}(a+b)\\0&0&3&a+c-2(a+b)\end{bmatrix}$$

$$\begin{matrix}\\\Rightarrow\\\frac{1}{3}R_3\end{matrix}\begin{bmatrix}1&0&\frac{5}{2}&\frac{1}{2}a-\frac{1}{2}b\\0&1&-\frac{1}{2}&\frac{1}{2}(a+b)\\0&0&1&-\frac{1}{3}a-\frac{2}{3}b+c\end{bmatrix}\begin{matrix}R_1-\frac{5}{2}R_3\\R_2+\frac{1}{2}R_3\\\Rightarrow\end{matrix}\begin{bmatrix}1&0&0&\frac{1}{2}a-\frac{1}{2}b-\frac{5}{2}(-\frac{1}{3}a-\frac{2}{3}b+c)\\0&1&0&\frac{1}{2}(a+b)+\frac{1}{2}(-\frac{1}{3}a-\frac{2}{3}b+c)\\0&0&1&-\frac{1}{3}a-\frac{2}{3}b+c\end{bmatrix}$$

Setting a = b = c = 0, it becomes clear that the set is linearly independent since the only solution to the system of equations A**x**=**0** is the trivial solution. Also, since the system above, A**x**=**v** for an arbitrary **v** is always consistent, S spans R^3. Because S is a linearly independent spanning set for R^3, **S is a basis for** $\mathbf{R^3}$.