1. Section 4.5, Page 314, question 4.

$$C=\begin{bmatrix}-6 & -1 & 2\\ 3 & 2 & 0\\ -14 & -2 & 5\end{bmatrix} \quad p(\lambda)=-(\lambda-1)^2(\lambda+1)$$

When $\lambda=1$, *a basis for the eigenspace can be found by solving for the nullspace of* $C-I$:

$$\begin{bmatrix}-7 & -1 & 2 & 0\\ 3 & 1 & 0 & 0\\ -14 & -2 & 4 & 0\end{bmatrix} \overset{R_1\leftrightarrow R_2}{\Rightarrow} \begin{bmatrix}3 & 1 & 0 & 0\\ -7 & -1 & 2 & 0\\ -14 & -2 & 4 & 0\end{bmatrix} \overset{\frac{1}{3}R_1}{\Rightarrow} \begin{bmatrix}1 & \frac{1}{3} & 0 & 0\\ -7 & -1 & 2 & 0\\ -14 & -2 & 4 & 0\end{bmatrix} \Rightarrow \begin{matrix}\\ R_2+7R_1\\ R_3+14R_1\end{matrix} \begin{bmatrix}1 & \frac{1}{3} & 0 & 0\\ 0 & \frac{4}{3} & 2 & 0\\ 0 & \frac{8}{3} & 4 & 0\end{bmatrix} \begin{matrix}\Rightarrow\\ \frac{3}{4}R_2\\ 3R_3\end{matrix}$$

$$\begin{bmatrix}1 & \frac{1}{3} & 0 & 0\\ 0 & 1 & \frac{3}{2} & 0\\ 0 & 8 & 12 & 0\end{bmatrix} \begin{matrix}R_1-\frac{1}{3}R_2\\ \Rightarrow\\ R_3-8R_2\end{matrix} \begin{bmatrix}1 & 0 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{3}{2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad x=\begin{bmatrix}\frac{1}{2}x_3\\ -\frac{3}{2}x_3\\ x_3\end{bmatrix}=x_3\begin{bmatrix}\frac{1}{2}\\ -\frac{3}{2}\\ 1\end{bmatrix} \text{ for arbitrary } x_3.$$

Hence, a basis for E_λ for Matrix C with $\lambda=1$ is $\begin{bmatrix}\frac{1}{2}\\ -\frac{3}{2}\\ 1\end{bmatrix}$. Therefore, the geometric multiplicity of $\lambda=1$ is 1 since this is the dimension of the eigenspace, while the algebraic multiplicity is 2 (since $\lambda=1$ is a repeated root of the characteristic function).

2. Section 4.5, Page 314, question 22. (Idempotent is defined in question 21.)

Given: $P\mathbf{x}=\lambda\mathbf{x},\quad \mathbf{x}\neq 0$ *and* $P=P^2$

$P(P\mathbf{x})=P(\lambda\mathbf{x})\Rightarrow P\mathbf{x}=P\lambda\mathbf{x}$

For the last equality to hold, λ can only be 0 or 1, since $\lambda=1$ gives $P\mathbf{x}=P\mathbf{x}$ and $\lambda=0$ is the eigenvalue for the eigenspace that is equivalent to the null space of the Matrix P. Any other scalar λ will leave us with no $\mathbf{x}$ vectors that will satisfy the equation.

3. Section 4.6, Page 324, question 20.

$$p(\lambda)=\begin{vmatrix}2-\lambda & 4\\ -2 & -2-\lambda\end{vmatrix}=((2-\lambda)(-2-\lambda))-((4)(-2))=\lambda^2-4+8=\lambda^2+4$$

$\lambda=\frac{\pm\sqrt{-4(1)(4)}}{2(1)}=\pm 2i=0+2i$ *and* $0-2i$

Let: $\lambda=2i$

$$\begin{bmatrix}2-2i & 4 & 0\\ -2 & -2-2i & 0\end{bmatrix} \qquad \frac{2}{2-2i}=\frac{2(2+2i)}{(2-2i)(2+2i)}=\frac{4+4i}{4-4i^2}=\frac{4+4i}{8}=\frac{1}{2}+\frac{1}{2}i$$

$$\overset{\Rightarrow}{R_2+\frac{2}{2-2i}R_1} \begin{bmatrix}2-2i & 4 & 0\\ 0 & -2-2i+2+2i & 0\end{bmatrix} \Rightarrow \begin{bmatrix}2-2i & 4 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$(2-2i)x_1=-4x_2\Rightarrow(1-i)x_1=-2x_2\Rightarrow(1-i)x_1=-2x_2 \qquad \frac{-2}{1-i}=\frac{-2(1+i)}{(1-i)(1+i)}=\frac{-2-2i}{1-i^2}=-1-i\ \therefore$

$x=\begin{bmatrix}(-1-i)x_2\\ x_2\end{bmatrix}=x_2\begin{bmatrix}-1-i\\ 1\end{bmatrix}$ *for arbitrary* x_2. *By theorem* 16 *and the characteristic function above,* $\overline{\lambda}$ *must also be an eigenvalue, with* $\overline{x}$ *as its eigenvector. Hence the eigenvalues and eigenvectors are*:

$\lambda=2i,\ -2i \quad x=x_2\begin{bmatrix}-1-i\\ 1\end{bmatrix},\ x_2\begin{bmatrix}-1+i\\ 1\end{bmatrix}$ *respectively for arbitrary* x_2.

4. (MAPLE) Section 4.6, Page 325, question 32.

```
> with(LinearAlgebra):
> A:=Matrix([[1,2,8],[8,4,9],[2,6,1]], datatype = float);
```

$$A := \begin{bmatrix} 1. & 2. & 8. \\ 8. & 4. & 9. \\ 2. & 6. & 1. \end{bmatrix}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 13.3768602100650185 + 0.\,I \\ -3.68843010503250522 + 2.84160362815091449\,I \\ -3.68843010503250522 - 2.84160362815091449\,I \end{bmatrix}$$

$$\begin{bmatrix} 0.418386816599859824 + 0.\,I & 0.699878732856667906 + 0.\,I & 0.699878732856667906 + 0.\,I \\ 0.788923140272879242 + 0.\,I & -0.106669529931524828 - 0.467136225725467812\,I & -0.106669529931524828 + 0.467136225725467812\,I \\ 0.450058607780602415 + 0.\,I & -0.383499182641769409 + 0.365381299750263456\,I & -0.383499182641769409 - 0.365381299750263456\,I \end{bmatrix}$$

Each eigenvalue is represented by an entry in the top matrix.

Its corresponding eigenvector is represented by the respective column in the second matrix. Ie. The first column of the second matrix is the eigenvector corresponding to the only real eigenvalue.