Diff Eq.

Fourier Series

Begin with the convergent series

\[ f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right) \]

Basic Facts:

1. The period of \( \cos \frac{m\pi x}{L} \) and \( \sin \frac{m\pi x}{L} \):

2. The period of \( f \):

3. Orthogonality:
4. Euler-Furier Formulas
Example 1: Let

\[ f(x) = \begin{cases} 
0 & -3 < x < -1 \\
1 & -1 < x < 1 \\
0 & 1 < x < 3 
\end{cases} \]

Suppose \( f(x+6) = f(x) \) \( [L = 3] \).

Plot \( f \) from \(-9\) to \(9\).

Determine the Fourier series.
Example 2:

\[ f(x) = -x \quad -2 \leq x < 2 \]

\[ f(x + 4) = f(x). \quad [L = 2] \]

Graph the function from -6 to 6.
Show the following:

\[ a_0 = 0 \]

\[ a_n = 0 \quad n = 1, 2, \ldots \]

\[ b_n = \frac{H}{n \pi} (-1)^n \]

\[ f(x) = \sum_{n=1}^{\infty} \frac{H}{n \pi} (-1)^n \sin \frac{n \pi x}{2} \]
Even and Odd Functions

\( f(x) \) is an even function if \( f(x) = f(-x) \).

Examples of even functions:

A. \( f(x) = x^2 \) \quad -2 \leq x \leq 2 \quad f(x) = f(x+4) \quad [\ell = 2] 

Graph of \( f(x) \):

\[ b_n = \int_{-2}^{2} f(x) \sin \frac{n\pi x}{2} \, dx = \int_{-2}^{2} x^2 \frac{\sin n\pi x}{2} \, dx = 0 \]
B. $f(x) = \text{constant}$ is even.

$f(x) = \cos ax$ is even (any constant $a$)

Graph of $f(x) = \cos ax$
Any cosine series is even.

3. Return to Example 1:

\[ f(x) = \begin{cases} 
0 & -3 < x < -1 \\
1 & -1 < x < 1 \\
0 & 1 < x < 3 
\end{cases} \]

\( f \) is even.

Recall the graph:

\[ \begin{array}{c}
-7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 \\
\end{array} \]

D. If \( h \) and \( g \) are both even, then \( h + g \) also is.
Odd Functions

$\text{f is an odd function iff } f(x) = -f(-x)$. 

Examples of odd functions:

A. $\sin a x$  any constant $a$

Graph of $\sin a x$

B. Fourier Sine Series
C. Recall Example 2:

\[ f(x) = -x \quad -2 \leq x < 2 \]

\[ f(x) = f(x + 4) \quad (l = 2) \]

Graph of \( f(x) \)

\[ f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} (-1)^n \sin \frac{n\pi x}{2} \]

D. If \( f \) and \( g \) are both odd so is \( h + g \).
**Diff Eq.**

**Convergence Theorem**

Suppose $f$ and $f'$ are piecewise continuous on $-L \leq x \leq L$ and we extend $f$ periodically by $f(x) = f(x+L)$. Then the Fourier series for $f(x)$ sums to the values

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] = \begin{cases} \frac{f(x)}{2} & \text{at a point of continuity} \\ \frac{f(x+)+f(x-)}{2} & \text{at a point of discontinuity} \end{cases}$$

**Example 1:**

$$f(x) = \begin{cases} 0 & -3 \leq x < -1 \\ 1 & -1 \leq x \leq 1 \\ 0 & 1 < x \leq 3 \end{cases}$$

$f(x) = f(x+6)$

[Graph of Sum of Fourier Series]

**Fourier Series for $f(x)$ is**
Example 2:

\[ f(x) = -x \quad -2 \leq x < 2 \]
\[ f(x+2) = f(x) \quad (L=2) \]

Fourier Series for \( f(x) \) is

Graph for Fourier Series for \( f(x) \) is

[Graph showing a periodic function with values at -6, -4, -2, 2, 4, 6.]