Differential Equations, Test 2 Review material

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March 28, 2002

This is intended as an overview, not the end-all be-all of the material you should be studying! All material covered in class/recitation is fair game for an exam. Any complaints such as ”This kind of material wasn’t on the review sheet” will be summarily ignored. You are adults, deal.

1 Mechanical Vibrations

You should:

• Be able to solve second order, linear, constant coefficient equations (homogeneous and non-homogeneous)

• Be able to solve any simple, periodic forcing problem for the standard spring mass system.

• Know what ”beats,” ”resonance, ” and ”steady state” solutions mean in the context of mechanical vibrations. Also be able to tell whether a system is: free, underdamped, critically damped, or overdamped.

• Be able to graph simple solutions to mechanical vibration problems. These graphs should take into account both the initial conditions, and the scales(frequency, amplitude) of the system.

• Know the variation of parameters formula for solving more complex non-homogeneous problems.
1.1 Example

Find the position of a mass suspended on a linear spring at any time \( t \) given that its position is governed by the following initial value problem:

\[
2y'' + 4y' + 16y = 4 \cos 5t \quad (1)
\]

\[
y(0) = 0 \quad (2)
\]

\[
y'(0) = 1 \quad (3)
\]

To solve this problem we first solve the homogenous problem to find the natural frequency of the system.

\[
2y'' + 4y' + 16y = 0 \quad (4)
\]

To solve this equation we assume \( y = e^{rt} \), then we plug this into the equation; divide out the common factor and we are left with the characteristic polynomial.

\[
2 \frac{d^2}{dt^2}(e^{rt}) + 4 \frac{d}{dt}(e^{rt}) + 16e^{rt} = 0 \quad (5)
\]

\[
2r^2e^{rt} + 4re^{rt} + 16e^{rt} = 0 \quad (6)
\]

\[
2r^2 + 4r + 16 = 0 \quad (7)
\]

Now we use the quadratic formula to find the values of \( r \) which satisfy this equation.

\[
r = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(16)}}{2(2)} \quad (8)
\]

\[
r = -1 \pm \sqrt{1 - 8} \quad (9)
\]

\[
r = -1 \pm i\sqrt{7} \quad (10)
\]

These roots are complex, thus the real parts get placed in exponentials and the imaginary parts go to sines and cosines. Our general homogenous solution will be given by:

\[
y_{homo} = C_1 e^{-t} \sin \sqrt{7}t + C_2 e^{-t} \cos \sqrt{7}t \quad (11)
\]

Now that we have our general homogeneous solution we can solve the forcing portion of the problem without any fear of resonance... (The forcing function( 4 cos 5t) has a different frequency than the homogeneous solution: 5 \( \neq \sqrt{7} \)).
To solve the forcing problem we use the method of undetermined coefficients. We assume: 
\[ y_{\text{force}} = A \cos 5t + B \sin 5t. \] We then take derivatives and plug into the differential equation.

\[
\begin{align*}
y_{\text{force}} &= A \cos 5t + B \sin 5t \quad (12) \\
y'_{\text{force}} &= -5A \sin 5t + 5B \cos 5t \quad (13) \\
y''_{\text{force}} &= -25A \cos 5t - 25B \sin 5t \quad (14)
\end{align*}
\]

plug into equation \[(15)\]

\[ 2y'' + 4y' + 16y = 4 \cos 5t \quad (16) \]

\[ 2(-25A \cos 5t - 25B \sin 5t) + 4(-5A \sin 5t + 5B \cos 5t) + 16(A \cos 5t + B \sin 5t) = 4 \cos 5t \quad (17) \]

Note: \((17),(18)\) are the same equation. Now we multiply out and group all the sine and cosine functions together...

\[ (-50A + 20B + 16A) \cos 5t + (-50B - 20A + 16B) \sin 5t = 4 \cos 5t \quad (19) \]

We split this into two equations, one for sine and the other for cosine, then we solve those two equations simultaneously for \(A, B\).

\[
\begin{align*}
-50A + 20B + 16A &= 4 \quad (20) \\
-50B - 20A + 16B &= 0 \quad (21) \\
-34A + 20B &= 4 \quad (22) \\
-34B - 20A &= 0 \quad (23)
\end{align*}
\]

Now solving \((22)\) for \(A\).

\[
\begin{align*}
-34A &= 4 - 20B \quad (24) \\
A &= \frac{20B - 4}{34} \quad (25)
\end{align*}
\]
Plug this into (23) and solve for $B$.

\begin{align*}
-34B - 20A &= 0 \quad (27) \\
-34B - 20\left(\frac{20B - 4}{34}\right) &= 0 \quad (28) \\
-(34)^2 B - 20(20B - 4) &= 0 \quad (29) \\
-(34)^2 B - 400B + 80 &= 0 \quad (30) \\
(34)^2 + 400)B &= 80 \quad (31) \\
B &= \frac{80}{(34)^2 + 400} \quad (32) \\
B &= \frac{80}{1556} \quad (33)
\end{align*}

Now use this to solve for $A$.

\begin{align*}
A &= \frac{20B - 4}{34} \quad (34) \\
A &= \frac{20\left(\frac{80}{1556}\right) - 4}{34} \quad (35) \\
A &= \frac{1600}{1556} - \frac{4}{34} \quad (36) \\
A &= \frac{1600}{1556*34} - \frac{1556*4}{1556*34} \quad (37) \\
A &= \frac{-4624}{52904} \quad (38) \\
A &\approx -\frac{3}{34} \quad (39)
\end{align*}

The reason these constants are so messy is that I came up with this problem from the top of my head. Test questions will not likely have algebra which is this cumbersome.

Now with the values of $A, B$ we can write down our non-homogeneous forcing solution:

\begin{align*}
y_{\text{force}} = \frac{-4624}{52904} \cos 5t + \frac{80}{1556} \sin 5t \quad (40)
\end{align*}

Now our final step is to combine the forcing solution with the homoge-
nous general solution and to solve for the constants $C_1, C_2$. 

$$Y = y_{homo} + y_{force}$$ (41)

$$Y = C_1 e^{-t} \sin \sqrt{7} t + C_2 e^{-t} \cos \sqrt{7} t + \frac{-4624}{52904} \cos 5t + \frac{80}{1556} \sin 5t$$ (42) (43)

Now we apply initial conditions...

$$Y(0) = C_1 e^0 \sin 0 + C_2 e^0 \cos 0 + \frac{-4624}{52904} \cos 0 + \frac{80}{1556} \sin 0$$ (44)

$$0 = C_2 + \frac{-4624}{52904}$$ (45)

$$C_2 = \frac{4624}{52904}$$ (46)

Now differentiate to apply the second initial condition (use product rule and so on...)

$$Y'(0) = C_1 e^0 \cos 0 - \sqrt{7} C_2 e^0 \cos 0 + 5 \cdot \frac{80}{1556} \cos 0$$ (47)

$$1 = C_1 - \frac{4624}{52904} + \frac{400}{1556}$$ (48)

$$C_1 = \frac{52904}{52904} + \frac{4624 \sqrt{7}}{52904} - \frac{400 \cdot 34}{52904}$$ (49)

$$C_1 = \frac{39304 + 4624 \sqrt{7}}{52904}$$ (50)

And we substitute these unpleasant values of $C_1, C_2$ into our general solution $Y(t)$ and we have solved this forcing problem.

$$Y = \frac{39304 + 4624 \sqrt{7}}{52904} e^{-t} \sin \sqrt{7} t + \frac{4624}{52904} e^{-t} \cos \sqrt{7} t + \frac{-4624}{52904} \cos 5t + \frac{80}{1556} \sin 5t$$ (51)

I have done this problem very explicitly so that there should be no questions about any step in the procedure. For test questions on vibrations, you should follow the same procedure as this problem, albeit without all of the messy algebra. (Algebraic mistakes are a possibility, if you see any please let me know.)

Now that we have the solution we can say some things about its structure. This is a damped forcing problem, so the solution consists
of two parts: transient and steady-state. The transient solution is the part of the solution which decays in time (If you take \( \lim_{t \to -\infty} \) the part which goes to zero is the transient part). The steady state is the part which sticks around for all time. In this case the homogenous solution is transient and the forcing solution is the steady state.

If we had to graph this solution we should draw a wave which starts at the origin with positive slope (see initial conditions) and given the complex nature of the constants, it should suffice to draw a wave whose vertical scale is less than one, and whose period approaches \( \frac{2\pi}{5} \) (\( 2\pi \) over the steady state frequency)

On your own you should look at some more problems, specifically undamped free vibrations, and resonant vibrations. Also remember that solutions can be converted to the \( R \cos (\omega t + \delta) \) form with some simple operations, you should be able to convert simple solutions into this form.

2 Fourier Series

You should:

- Understand the derivation of Fourier Series, periodicity, orthogonality and so on..
- Memorize the Euler-Fourier formulas for computing the Fourier Coefficients.
- Be able to compute the Fourier series of simple functions (i.e. find the fourier coefficients and write them in a relatively compact fashion)
- Be able to identify "Even" and "Odd" functions, and use the properties of even and odd”ness” to simplify the computation of Fourier Series.
- Be able to compute the Fourier series of a function defined on \((0, L)\) by extending said function either evenly or oddly on to the interval \((-L, 0)\).
- Be able to graph the functions to which the Fourier series actually converges(i.e. draw graphs which explicitly note that the Fourier
series converges to the mid-points of any jump discontinuities, and converge to the periodic extension of \((-L, L)\)).

2.1 Example

Find the Fourier Series for \(f(x) = x^2 + 1\) on the interval \((0, 2)\) by using the Odd extension of this function to \((-2, 0)\).

Since we want to find the Fourier Sine series of this function (Fourier Sine series, because we’re looking at the odd extension). We only need to find the coefficients \(b_n\), which make up this series.

We know \(b_n\) is given by the Euler-Fourier formulas, but we’re only working on the half interval \((0, L)\) (Note: \(L = 2\) in this case) so we use the Modified Euler-Fourier formula for \(b_n\).

\[
b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx
\]  

\[
b_n = \frac{2}{L} \int_0^L (x^2 + 1) \sin \frac{n\pi x}{2} dx
\]  

We compute the value of this integral by using integration by parts twice.

\[
b_n = \int_0^2 (x^2 + 1) \sin \frac{n\pi x}{2} dx
\]

\[
b_n = -(x^2 + 1)(\frac{2}{n\pi} \cos \frac{n\pi x}{2})\mid_0^2 + \frac{2}{n\pi} \int_0^2 2x \cos \frac{n\pi x}{2} dx
\]

\[
b_n = -(x^2 + 1)(\frac{2}{n\pi} \cos \frac{n\pi x}{2})\mid_0^2 + \frac{2}{n\pi} [(2x)(\frac{2}{n\pi} \sin \frac{n\pi x}{2})\mid_0^2 - \frac{2}{n\pi} \int_0^2 2 \sin \frac{n\pi x}{2} dx]
\]

\[
b_n = -(x^2 + 1)(\frac{2}{n\pi} \cos \frac{n\pi x}{2})\mid_0^2 + \frac{2}{n\pi} [0 + -\frac{2}{n\pi} \int_0^2 2 \sin \frac{n\pi x}{2} dx]
\]

\[
b_n = -(x^2 + 1)(\frac{2}{n\pi} \cos \frac{n\pi x}{2})\mid_0^2 + \frac{2}{n\pi} [-\frac{8}{n^3\pi^3} \cos \frac{n\pi x}{2}]
\]

\[
b_n = -\frac{4}{n\pi} \cos n\pi + \frac{2}{n\pi} \cos \frac{n\pi x}{2}\mid_0^2 + \frac{8}{n^3\pi^3} \cos \frac{n\pi x}{2}\mid_0^2
\]

\[
b_n = \frac{4}{n\pi} (-1)^n + \frac{2}{n\pi} + \frac{8}{n^3\pi^3} [\cos n\pi - 1]
\]

\[
b_n = \begin{cases} 
\frac{6}{n\pi} + \frac{16}{n\pi^3} & \text{for } n \text{ odd} \\
\frac{4}{n^2\pi^2} & \text{for } n \text{ even}
\end{cases}
\]
With these values for $b_n$ we can write down the Fourier Sine series for $f(x)$ (I will omit the actual values of $b_n$ as they are cumbersome)

$$x^2 + 1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \text{ for } 0 < x < 2 \quad (54)$$

$$b_n = \begin{cases} 
\frac{6}{n\pi} + \frac{16}{n^3\pi^3} & \text{for } n \text{ odd} \\
\frac{2}{n\pi} & \text{for } n \text{ even} 
\end{cases} \quad (55)$$

At the endpoints of this interval the periodic extension of this function has jump discontinuities, thus the Fourier series only converges to the function on the open interval $(0, 2)$. At the endpoints it converges to the midpoint of the discontinuity. (Note: one would find a totally different Fourier series if one extended this function evenly to find a cosine series rather than extending oddly as we did)

3 Partial Differential Equations

You should:

- Understand and be able to perform separation of variables on simple partial differential equations.

- Understand the eigenvalue problem, be able to pose and solve an eigenvalue problem given a partial differential equation.

- Feel comfortable with the solutions to the Heat and Wave equations (i.e. be able to compute particular solutions given the general solution for a particular set of boundary conditions)

- Understand the differences between different sets of boundary conditions (homogenous, non-homogeneous, and insulated ends) and their physical interpretations and what effects they have on the solution to the heat equation.

3.1 Example

For purposes of an example, we’re going to work through solving the heat equation with non-homogenous (but constant) boundary conditions.
Say a 2 meter bar of unobtainium is heated to a uniform temperature of 50°C then one end of the bar is placed against a conductive kiln wall at a constant temperature of 300°C and the other end is placed in an ice bath at 0°C. We want to find an expression for the temperature in the bar at any point \( x \) for any time after the bar is in place.

The information given in the preceding paragraph gives us the following heat conduction problem:

\[
    u_{xx} = u_t \text{ for } 0 < x < 2 \text{ and } t > 0
\]

\[
    u(0, t) = 0 \quad u(2, t) = 300 \quad u(x, 0) = 50
\]

(NOTE: Unobtainium has a diffusivity constant of \( \alpha = 1 \) in any units you like, so no \( \alpha \) appears in this problem)

The first thing we need to do in this problem is render the boundary conditions homogeneous, to do that we propose that our solution is a linear combination of a “steady-state” solution, \( w(x, t) \) and a “transient” solution, \( v(x, t) \).

\[
    u(x, t) = w(x, t) + v(x, t) \tag{56}
\]

To find out what our steady state solution is, we consider what happens in the rod after a long time. We would expect that after a long time, the rod will settle to some equilibrium solution where nothing changes with time.

We want the steady state solution to satisfy the non-homogeneous boundary conditions, and to satisfy the following reduced differential equation: \( w_{xx} = 0 \). This is just the original differential equation with all of the time dependence removed (nothing is changing with time).

This equation is easy to solve and its solution is simply a line:

\[
    w = Ax + B \tag{57}
\]

The boundary conditions force us to pick:

\[
    B = 0 \tag{58}
    \]

\[
    A = 150 \tag{59}
\]

So our steady state solution is:

\[
    w(x) = 150x \tag{60}
\]
Now we have a steady state solution, so we should expect our transient solution to satisfy zero conditions at both of the boundaries (we wouldn’t want the transient solution to add anything at the boundaries because the steady state solution takes care of those.)

In addition to homogenous boundary conditions, we want the transient solution to satisfy the initial condition, HOWEVER, the initial condition has changed slightly... since we have this steady state solution hanging around, we want our transient solution to satisfy the following condition:

\[ v(x, 0) = 50 - w(x) \] (61)

We want this modified condition because of the following argument:

\[ u(x, t) = w(x) + v(x, t) \] (by assumption) (62)
\[ u(x, 0) = 50 \] (initial condition) (63)
\[ u(x, 0) = w(x) + v(x, 0) \] (set \( t = 0 \)) (64)
\[ v(x, 0) = u(x, 0) - w(x) \] (algebra) (65)
\[ v(x, 0) = 50 - w(x) \] (identity replacement) (66)

Ok so now we have to solve the following problem for our transient solution \( v(x, t) \).

\[ v_{xx} = v_t \] for \( 0 < x < 2 \) and \( t > 0 \)
\[ v(0, t) = 0 \quad v(2, t) = 0 \quad v(x, 0) = 50 - 150x \]

Now we use separation of variables to split this PDE into two odes.

Let: \( v(x, t) = X(x)T(t) \) (67)
\[ \frac{\partial^2}{\partial x^2} X T = \frac{\partial}{\partial t} X T \] (68)
\[ X''T = T'X \] (69)
\[ \frac{X''}{X} = \frac{T'}{T} \] (70)

Now we say the standard arguments, ”\( x, t \) are independent variables...blah...blah...blah” and set both of these ratios equal to a constant \(-\lambda\) which is yet to be
Here (72) and (74) are ordinary differential equations which we can solve to find values of $\lambda$ which give us non-trivial solutions. We now solve the eigenvalue problem for the $X$ equation to find valid values for $\lambda$. (We solve the $X$ equation because the initial conditions on $T$ depend on $x$, while the boundary conditions on $X$ are totally devoid of $t$-dependence.)

Before we solve the eigenvalue problem, we need to take our boundary conditions on $v(x, t)$ and translate them into conditions on $X(x)$. This is relatively straightforward, the conditions simply inherit their exact character from the conditions on $v$. (If you want to see the argument for this again, look in the book). After translation we are left with the following problem for $X$. THIS IS THE CORRECTLY POSED EIGENVALUE PROBLEM FOR HOMOGENEOUS BOUNDARY CONDITIONS.

$$X'' + \lambda X = 0 \quad (75)$$
$$X(0) = 0 \quad (76)$$
$$X(2) = 0 \quad (77)$$

Now we make assumptions about the sign of $\lambda$ to check what types of solutions to this equation will give us non-trivial solutions. Once again, this whole argument is in the book, if you check $\lambda < 0$ and $\lambda = 0$ they give only the trivial solution for $X$ (in this case only, different things happen for different boundary conditions). So we check $\lambda > 0$

$$\lambda = \mu^2 \quad (78)$$
$$X'' + \mu^2 X = 0 \quad (79)$$

This is a second order linear ODE with constant coefficients, we know how to solve this: (1)assume $X = e^{\mu t}$, (2)substitute and get the characteristic polynomial. (3) Find the roots of the characteristic polynomial. (4) write down the general solution. Skipping everything, the
general solution becomes:

\[ X = A \cos \mu x + B \sin \mu x \]  \hfill (80)

Now we apply the boundary conditions.

\[ X(0) = A \cos(0) + B \sin(0) \]  \hfill (81)

\[ 0 = A \]  \hfill (82)

\[ X(2) = B \sin \mu 2 \]  \hfill (83)

Now if we pick \( B = 0 \) we have the trivial solution, but if we make \( \sin(\mu 2) = 0 \) then we have non-trivial solutions. SO choose \( \mu = \frac{n \pi}{2} \) and we have non-trivial solutions.

\[ \mu = \frac{n \pi}{2} \]  \hfill (84)

\[ \lambda = \mu^2 \]  \hfill (85)

\[ \lambda = \frac{n^2 \pi^2}{4} \text{ for } n = 1, 2, 3 \ldots \]  \hfill (86)

\[ X_{\lambda_n} = B_{\lambda_n} \sin \frac{n \pi x}{2} \]  \hfill (87)

Now we have a \( X \) solution for each one of these \( \lambda \)'s so we need the corresponding \( T \) solution. This can be found easily by solving (74) using direct integration or an integrating factor. Now for each \( \lambda \) we get the following solution:

\[ v_{\lambda_n} = X_{\lambda_n} T_{\lambda_n} \]  \hfill (88)

\[ v_{\lambda_n} = B_n \sin \frac{n \pi x}{2} e^{-\frac{n^2 \pi^2}{4} t} \]  \hfill (89)

To write down our general solution we simply take a linear combination of all of the solutions we’ve found (since there are an infinite number of \( \lambda \)'s we will have an infinite series.)

\[ v_{\text{gen}}(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{2} e^{-\frac{n^2 \pi^2}{4} t} \]  \hfill (90)

Now we are in possession of the general solution, and it only remains to find the specific solution. The only condition we have left to satisfy
from the original problem is the initial condition so:

\[
v(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} e^{0} \quad (91)
\]
\[
v(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} \quad (92)
\]
\[
50 - 150x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} \quad (93)
\]

This is simply a Fourier series expression, so we evaluate the \( B_n \) using the modified Euler-Fourier formulas for an oddly extended function and then we will have possession of the particular transient solution.

I will not evaluate the \( B_n \) explicitly here, you should be able to do that by following the example in the Fourier series section of this review sheet.

Going back to our original problem we said that:

\[
u(x, t) = w(x) + v(x, t) \quad (94)
\]

So our final solution for \( u(x, t) \) will be the following:

\[
u(x, t) = 150x + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} e^{-\frac{n^2\pi^2 x^2}{4} t} \quad (95)
\]

For appropriate values of the \( B_n \).