This is a CLOSED BOOK test.

1. Please answer all questions, showing your work in detail and giving reasons where appropriate.
2. Collaboration with other students is NOT permitted.
3. Be sure you have 7 test pages for this test.
4. Point allocations for each question are indicated. Plan your time accordingly. The total number of points is 100.

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1. **(a) (8 points)** Consider the matrix \( A = \begin{pmatrix} 0 & -6 \\ 1 & 0 \end{pmatrix} \). Find the eigenvalues of \( A \).

**(b) (8 points)** The eigenvalues of \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) are \( \lambda_1 = \frac{3 + \sqrt{5}}{2} \) and \( \lambda_2 = \frac{3 - \sqrt{5}}{2} \). Find the eigenvectors corresponding to each of the eigenvalues.
1. (c) (8 points) The eigenvalue, eigenvector pairs for \( A = \begin{pmatrix} 0 & 6 \\ 1 & 0 \end{pmatrix} \) are:

\[
\begin{align*}
\lambda_1 &= \sqrt{6}, \quad \xi_1 = \begin{pmatrix} 1 \\ 1/\sqrt{6} \end{pmatrix} \\
\lambda_2 &= -\sqrt{6}, \quad \xi_2 = \begin{pmatrix} 1 \\ -1/\sqrt{6} \end{pmatrix}
\end{align*}
\]

Write down the general solution for \( \tilde{x}' = A \tilde{x} \). What kind of critical point is \( \tilde{x} = 0 \)?

2. The general solution is given for each linear equation below. Make a phase plane plot and say what kind of critical point zero is.

(a) (8 points) \( \tilde{x}' = \begin{pmatrix} 1 & 9 \\ 1 & 1 \end{pmatrix} \tilde{x}, \quad \tilde{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 1/3 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} \)
2. (b) (8 points) $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x, \quad x(t) = c_1 \left( \frac{1}{\sqrt{5}} \cos t \sin (t + \alpha) \right) + c_2 \left( -\frac{1}{\sqrt{5}} \sin t \cos (t + \alpha) \right)$

where $0 < \alpha < \frac{\pi}{2}$, tan $\alpha = 2$.

2. (c) (10 points) The two complex solutions for $x' = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} x$ are $e^{3t} e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $e^{3t} e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Find the real solutions and write the general solution in terms of these real solutions.
3. Consider the competing species problem

\[ \frac{dx}{dt} = x(2 - x - y) \]
\[ \frac{dy}{dt} = y \left( 1 - y - \frac{1}{2} x \right) \]

(a) (8 points) Two of the critical points are \( \bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( \bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).
Find the other critical points.

(b) (8 points) For the critical point \( \bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), find the linearized equation.
3. (c) (8 points) We can rewrite our system of equations as

\[
\frac{dx}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} -x(x + y) \\ -y(y + x/2) \end{pmatrix}
\]

Justify that this is an almost linear equation for the critical point \( \tilde{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

3. (d) (8 points) Determine what kind of critical point \( \tilde{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is in part (c) by finding the solution of the linearized equation.
4. (a) (10 points) Determine the periodic solutions of an autonomous equation written in polar coordinates as

\[ \frac{dr}{dt} = -r(2 - r)(4 - r), \]
\[ \frac{d\theta}{dt} = -1. \]

(b) (8 points) Write down your instructor's name.