This is a **CLOSED BOOK** test.

1. Please answer all questions, showing your work in detail and giving reasons where appropriate.
2. Collaboration with other students is **NOT** permitted.
3. Be sure you have 7 test pages for this test.
4. Point allocations for each question are indicated. Plan your time accordingly. The total number of points is 100.

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1. (a) (8 points) Consider the matrix \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \). Find the eigenvalues of \( A \).

(b) (8 points) The eigenvalues of \( A = \begin{pmatrix} 0 & 6 \\ 1 & 0 \end{pmatrix} \) are \( \lambda_1 = \sqrt{6}, \; \lambda_2 = -\sqrt{6} \). Find the eigenvectors corresponding to each of the eigenvalues.
1. (c) (8 points) The eigenvalue, eigenvector pairs for \( A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \) are:

\[
\lambda_1 = \frac{5}{2}, \; \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \lambda_2 = \frac{3}{2}, \; \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]

Write down the general solution for \( \mathbf{x}' = A \mathbf{x} \). What kind of critical point is \( \mathbf{x} = 0 \)?

2. The general solution is given for a linear equation in parts (a) and (b). In each part make a phase plane plot and say what kind of critical point zero is.

(a) (8 points) \( \mathbf{x}' = \begin{pmatrix} +1 & +1 \\ -1 & +1 \end{pmatrix} \mathbf{x} \), \( \mathbf{x}(t) = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \)
2. (b) (8 points) \( \dot{x} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x, \quad x(t) = c_1 \begin{pmatrix} \cos t \\ \frac{1}{\sqrt{5}} \sin (t + \alpha) \end{pmatrix} + \\
c_2 \begin{pmatrix} \sin t \\ -\frac{1}{\sqrt{5}} \cos (t + \alpha) \end{pmatrix} \) where \( 0 < \alpha < \frac{\pi}{2}, \tan \alpha = 2. \)

2. (c) (10 points) The two complex solutions for \( x' = \begin{pmatrix} +2 & -1 \\ +1 & +2 \end{pmatrix} x \)

are \( e^{2t} e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} \) and \( e^{2t} e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix} \). Find the real solutions and write the general solution in terms of these real solutions.
3. Consider the competing species problem

\[ \frac{dx}{dt} = x \left( \frac{3}{2} - x - y \right) \]
\[ \frac{dy}{dt} = y \left( 1 - y - \frac{1}{2} x \right) \]

(a) (8 points) Two of the critical points are \[ x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \] and \[ x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]. Find the other critical points.

(b) (8 points) For the critical point \[ x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \], write the linearized equations.
3. (c) (8 points) We can rewrite our system of equations as

\[
\frac{dx}{dt} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{pmatrix} \tilde{x} + \begin{pmatrix} -x(x+y) \\ -y(y+x/2) \end{pmatrix}
\]

Justify that this is an almost linear equation for the critical point \( \tilde{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

3. (d) (8 points) Determine what kind of critical point \( \tilde{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is in part (c) by finding the solution of the linearized equation.
4. (a) (10 points) Determine the periodic solutions of an autonomous
   equation written in polar coordinates as
   \[ \frac{dr}{dt} = -r(1 - r)(3 - r), \quad \frac{d\theta}{dt} = 1. \]

(b) (8 points) Write your instructor’s name.