This is a **CLOSED BOOK** test.

1. Please answer all questions, showing your work in detail and giving reasons where appropriate.

2. Collaboration with other students is **NOT** permitted.

3. Be sure you have 6 test pages for this test.

4. Point allocations for each question are indicated. Plan your time accordingly. The total number of points is 100.

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1. A mass weighing 4 pounds stretches a spring 3 inches. The mass is in a viscous medium that exerts a viscous resistance of 1 pound when the mass has a velocity of 4 feet/sec. The mass initially has no displacement with initial velocity 1 foot/sec.

(a) (8 points) Write the differential equation and initial conditions for this problem, finding the mass and spring constants (using feet as the length scale);

\[ mg = \gamma \Rightarrow m = \frac{\gamma}{g^2} = \frac{1}{8} \]

\[ mg = \gamma = k \cdot \frac{1}{k} \Rightarrow k = 16 \]

\[ \gamma \frac{du}{dt} = \gamma, \; \gamma = 1 \Rightarrow \gamma = \frac{1}{t} \]

\[ \frac{1}{8} u'' + \frac{1}{7} u' + 16 u = 0 \]

\[ \begin{cases} u(0) = 0 \\ u'(0) = 1 \end{cases} \]

(b) (12 points) Find the solution for the problem in part (a);

\[ u'' + 2u' + 128u = 0 \]

\[ u = e^{rt}, \quad u' = re^{rt}, \quad u'' = r^2e^{rt} \]

\[ (r^2 + 2r + 128)e^{rt} = 0 \]

\[ r = -1 \pm \sqrt{1 - 128} = -1 \pm \sqrt{-127} \]

\[ e^{-t}e^{\sqrt{127}i t} = e^{-t}[\cos\sqrt{127}t + \sin\sqrt{127}t] \]

\[ e^{-t}e^{-\sqrt{127}i t} = e^{-t}[\cos\sqrt{127}t - \sin\sqrt{127}t] \]

\[ u_1 = e^{-t}\sin\sqrt{127}t, \quad u_2 = e^{-t}\cos\sqrt{127}t \]
\[ u = c_1 e^{-t} \sin(127t) + c_2 e^{-t} \cos(127t) \]

\[ u(0) = 0 \Rightarrow c_2 = 0 \Rightarrow u = c_1 e^{-t} \sin(127t) \]

\[ u'(0) = 1. \]

\[ u' = c_1 \left[ -e^{-t} \sin(127t) + 127 e^{-t} \cos(127t) \right] \]

\[ u'(0) = 1 \Rightarrow 1 = 127 c_1 \]

\[ \Rightarrow u = \frac{1}{127} e^{-t} \sin(127t) \]

(c) (10 points) Graph the solution making certain to graph the correct initial condition.
(2) Let \( f : [-2, 2) \rightarrow \mathbb{R} \)

\[
f(x) = \begin{cases} 
2, & -2 \leq x < -1 \\
1, & -1 \leq x < 0 \\
1, & 0 \leq x < 1 \\
2, & 1 \leq x < 2 
\end{cases}
\]

(a) (12 points) Is \( f(x) \) on \([-2, 2]\) even, odd, or neither? Extend \( f(x) \) periodically to \([-4, 4]\). Plot \( f(x) \) on \([-4, 4]\).

We see that \( f(-x) = f(x), \forall x > 0 \), then \( f(x) \) is even.

Here \( L = 2 \), and then the period \( T = 2L = 4 \), then the extending \( f \) to \([-4, 4]\), we use: \( f(x + 4) = f(x), \forall x \in \mathbb{R} \).

The plot of \( f \) on \([-4, 4]\):

(b) (15 points) Find the corresponding Fourier series for \( f \). If \( f(x) \) is even or odd, take advantage of this in finding the Fourier coefficients \( a_m \) and \( b_m \).

Since \( f(x) \) is even,

\[
b_m = 0, \quad m = 1, 2, \ldots
\]

\[
a_m = \frac{2}{L} \int_0^L f(x) \cos \frac{m\pi x}{L} \, dx = \int_0^1 f(x) \cos \frac{m\pi x}{2} \, dx + \int_1^2 f(x) \cos \frac{m\pi x}{2} \, dx = \frac{2}{m\pi} \sin \frac{m\pi x}{2} \bigg|_0^1 + \frac{1}{m\pi} \sin \frac{m\pi x}{2} \bigg|_1^2 = \frac{2}{m\pi} \sin \frac{m\pi}{2} - \frac{1}{m\pi} \sin \frac{m\pi}{2} = -\frac{2}{m\pi} \sin \frac{m\pi}{2}
\]

\[a_0 = \int_0^L f(x) \, dx = \int_0^1 1 \, dx + \int_1^2 2 \, dx = 1 + 2 = 3\]

Then the Fourier series of \( f \):

\[
f(x) = \frac{3}{2} - \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin \frac{m\pi x}{2} - \cos \frac{m\pi x}{2}
\]
(c) (8 points) Sketch the graph of the function to which the Fourier series converges on \([-2, 2]\).

Using the Fourier convergence theorem, we can plot the function to which the previous Fourier series converges:

Here \(x\) represents a point to which the Fourier series converges at a discontinuity point.

(3)(a) (15 points) Use separation of variables to split the following partial differential equation into two ordinary differential equations. (Please proceed in a logical manner and include all of the standard arguments)

\[ 16u_{xx} = u_{tt} \]

Assume \( U(x,t) = X(x)T(t) \)

\[ 16 \frac{\partial^2}{\partial x^2}(XT) = \frac{\partial^2}{\partial t^2}(XT) \]

\[ 16X''T = XT'' \]

\( X'' = \frac{T''}{16T} = -\lambda \)

\[ X'' + \lambda X = 0 \quad T'' + 16\lambda T = 0 \]
(b) (15 points) Given the following general solution and initial condition to a particular heat conduction problem, find the sequence of $c_n$'s which give the specific solution

$$u_{gen}(x, t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2 \pi^2 t}{4}} \sin \left( \frac{n \pi x}{2} \right), 0 < x < 2, \quad t > 0,$$

$$u(x, 0) = 2x + 1, \quad 0 < x < 2.$$

$$C_n = \frac{2}{L} \int_0^L (2x+1) \sin \left( \frac{n \pi x}{L} \right) dx = \int_0^L (2x+1) \sin \left( \frac{n \pi x}{L} \right) dx$$

Integrate by Parts...

$$= 2x+1 \left( \frac{2}{n \pi} \cos \left( \frac{n \pi x}{L} \right) \right)_0^L + \frac{2}{n \pi} \int_0^L 2 \cos \left( \frac{n \pi x}{L} \right) dx$$

$$C_n = \frac{2}{n \pi} - \frac{10}{n \pi} (-1)^n + \text{box}$$

(c) (5 points) What type of boundary conditions would give rise to the general solution in (b)? (fixed temperature, insulated ends, etc.). Give a mathematical expression for the boundary conditions.

Fixed $0^\circ C$ temperature gives a sine series solution. (homogeneous. B.C.5)

$U(0, t) = U(2, t) = 0$