This is a **CLOSED BOOK** test.

1. Please answer all questions, showing your work in detail and giving reasons where appropriate.

2. Collaboration with other students is **NOT** permitted.

3. Be sure you have 6 test pages for this test.

4. Point allocations for each question are indicated. Plan your time accordingly. The total number of points is 100.

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1. Solve the following equation by finding an explicit form of the solution $y(x)$ in:

(a) (10 points)

$$\frac{d}{dx} y = (1 - 2x)y^2, \quad y(0) = -\frac{1}{6}.$$  

\[
\frac{dy}{dx} = \frac{1 - 2x}{y^2} \Rightarrow \frac{dy}{y^2} = \frac{1 - 2x}{x^2} \Rightarrow \int \frac{dy}{y^2} = \int \frac{1 - 2x}{x^2} dx
\]

\[
-\frac{1}{y} = x - x^2 + c \Rightarrow y(0) = -\frac{1}{6} \Rightarrow c = \frac{1}{6} \]

\[
\Rightarrow y = \frac{1}{x^2 - x - \frac{1}{6}}
\]

(b) (10 points)

$$\frac{d}{dt} y = (t + 1)y, \quad y(0) = 3.$$  

\[
\text{3 pts.} \quad \frac{dy}{dt} = (t + 1)
\]

\[
\ln |y| = \frac{t^2}{2} + t + c
\]

\[
y = c e^{\frac{t^2}{2} + t}
\]

\[
y(0) = 3 \Rightarrow c = 3 \Rightarrow y = e^{\frac{t^2}{2} + t}
\]
2. A particular population of manatees is modeled by the following logistic equation.

\[ \frac{dm}{dt} = -\frac{1}{10}(1 - \frac{M}{20})(1 - \frac{M}{3000})M \]

(a) (10 points) Draw a stability diagram \( \frac{dm}{dt} \) vs \( M \) for this autonomous differential equation.

(b) (10 points) Determine all equilibrium solutions.

*equilibrium solutions are the zeros of the right hand side of the equation.*

\[ M = 0 \]
\[ M = 20 \]
\[ M = 3000 \]
2. (c) (10 points) If this population started with 10 manatees describe the evolution of this population (i.e. what would happen as \( t \to \infty \))

\[
\frac{dM}{dt} > 0
\]

\[
\frac{dM}{dt} < 0
\]

Population goes to zero.

3. (a) (10 points) Find two linearly independent solutions, \( y_1, y_2 \), of
\[ y'' + y' + 2y = 0; \]

\[
y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt} \]

want \( y'' + y' + 2y = (r^2 + r + 2)e^{rt} = 0 \)

\[
(r^2 + r + 2) = 0
\]

\[
r = \frac{-1 \pm \sqrt{1-4 \cdot 2}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}
\]

\[
e^{-\frac{1}{2}t}[\cos(\frac{\sqrt{7}}{2}t) + i\sin(\frac{\sqrt{7}}{2}t)] = e^{-\frac{1}{2}t}e^{i\frac{\sqrt{7}}{2}t} \]

\[
e^{-\frac{1}{2}t}e^{-\frac{1}{2}t} = e^{-\frac{3}{2}t} \]

\[
e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{7}}{2}t\right) = e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{7}}{2}t\right)
\]

\[
e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{7}}{2}t\right) = e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{7}}{2}t\right)
\]

\[
\Rightarrow y_1 = e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{7}}{2}t\right) \quad y_2 = e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{7}}{2}t\right)
\]
3. (b) (10 points) Using your answer from part (a) find the solution of \( y'' + 4y = 0 \), \( y(0) = 0 \), \( y'(0) = 1 \);

\[
y(t) = c_1 e^{-\frac{1}{2} t} \cos \frac{\sqrt{7}}{2} t + c_2 e^{-\frac{1}{2} t} \sin \frac{\sqrt{7}}{2} t
\]

\( y'(0) = 0 \) \( \Rightarrow \ c_1 = 0 \)

\[
y(t) = c_2 \left( -\frac{1}{2} e^{-\frac{1}{2} t} \sin \frac{\sqrt{7}}{2} t + \frac{\sqrt{7}}{2} c_2 e^{-\frac{1}{2} t} \cos \frac{\sqrt{7}}{2} t \right)
\]

\( y'(0) = 1 \) \( \Rightarrow \ \frac{\sqrt{7}}{2} c_2 = 1 \) \( \Rightarrow \ c_2 = \frac{2}{\sqrt{7}} \)

\[
y(t) = \frac{2}{\sqrt{7}} e^{-\frac{1}{2} t} \sin \frac{\sqrt{7}}{2} t
\]

(c) (10 points) Graph the solution you obtain in part (b), showing the behavior at \( t = 0 \) and as \( t \to \infty \).
4. The two linearly independent solutions of \( y'' + 7y = 0 \) are \( \sin \sqrt{7}t, \cos \sqrt{7}t \).

(a) (10 points) Write down the variation of parameter formula for finding a solution of \( y'' + 7y = \sin^2 t \), evaluating the Wronskian in this formula.

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1
\]

\( y_1 = \sin \sqrt{7}t \), \( y_2 = \cos \sqrt{7}t \), \( y_1 y'_2 - y_2 y'_1 = -\sqrt{7} (\cos^2 \sqrt{7}t + \sin^2 \sqrt{7}t) = -\sqrt{7} \)

\( y(t) = -y_1(t) \int_0^t \frac{y_2(s) g(s)}{W(y_1, y_2)(s)} \, ds + y_2(t) \int_0^t \frac{y_1(s) g(s)}{W(y_1, y_2)(s)} \, ds \)

\( = -\sin \sqrt{7}t \int_0^t \frac{\cos \sqrt{7}s \sin^2 s}{-\sqrt{7}} \, ds + \cos \sqrt{7}t \int_0^t \frac{\sin \sqrt{7}s \sin^2 s}{-\sqrt{7}} \, ds \)

(b) (10 points) Use your variation of parameters formula to write down the general solution of \( y'' + 7y = \sin^2 t \)

\( y(t) + c_1 y_1 + c_2 y_2 \)

\( = -\sin \sqrt{7}t \int_0^t \frac{\cos \sqrt{7}s \sin^2 s}{-\sqrt{7}} \, ds + \cos \sqrt{7}t \int_0^t \frac{\sin \sqrt{7}s \sin^2 s}{-\sqrt{7}} \, ds \)

\( + c_1 \sin \sqrt{7}t + c_2 \cos \sqrt{7}t \)