This is a **CLOSED BOOK** test.

1. Please answer all questions, showing your work in detail and giving reasons where appropriate.

2. Collaboration with other students is **NOT** permitted.

3. Be sure you have 6 test pages for this test.

4. Point allocations for each question are indicated. Plan your time accordingly. The total number of points is 100.

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1. Solve the following equation by finding an explicit form of the solution \( y(x) \) in:
(a) (10 points)
\[
\frac{d}{dx} y = -\frac{y x}{x^2 + 1}, \quad y(0) = 1.
\]
\[
\frac{dy}{dx} = \frac{-x}{y(x^2 + 1)}
\]
\[
\ln |y| = -\frac{1}{2} \ln |x^2 + 1| + C
\]
\[
y = \frac{C}{(x^2 + 1)^{1/2}}
\]
\[
y(0) = 1 \quad \Rightarrow \quad C = 1
\]
\[
y = \frac{1}{(x^2 + 1)^{1/2}}
\]

(b) (10 points)
\[
\frac{d}{dt} y = (t + 1)y, \quad y(0) = 3.
\]
\[
\left(\frac{dy}{dt} = t + 1 \right)
\]
\[
\ln |y| = \frac{t^2}{2} + t + C
\]
\[
y = e^{\frac{t^2}{2} + t + C}
\]
\[
y(0) = 3 \quad \Rightarrow \quad C = 3
\]
\[
y = 3e^{\frac{t^2}{2} + t}
\]
2. A certain valley contains a population of 3-toed sloths which is modeled by the following logistic equation.

\[
\frac{dS}{dt} = -\frac{1}{2} \left( 1 - \frac{S}{200} \right) \left( 1 - \frac{S}{630} \right)
\]

(a) (10 points) Draw a stability diagram \(\frac{dS}{dt} vs S\) for this autonomous differential equation.

(b) (10 points) Determine all equilibrium solutions.

\[
S = 200 \\
S = 630
\]

The equilibrium solutions are the zeros of the right hand side.
(c) (10 points) If this population started with 250 sloths describe the evolution of this population (i.e. what would happen as $t \to \infty$.)

$$\frac{ds}{dt} > 0$$

The population goes to 630.

3. (a) (10 points) Find two linearly independent solutions, $y_1, y_2$, of $y'' + 6y' + 8y = 0$;

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}$$

Want $y'' + 6y' + 8y = (r^2 + 6r + 8)e^{rt} = 0$

$$r^2 + 6r + 8 = 0$$

$$(r + 4)(r + 2) = 0 \quad r = -4, r = -2$$

$$y_1 = e^{-4t}, \quad y_2 = e^{-2t}$$
(b) (10 points) Using your answer from part (a) find the solution of
\[ y'' + 6y' + 8y = 0, y(0) = 3, y'(0) = 1; \]
\[ y = c_1 e^{-4t} + c_2 e^{-2t} \quad y' = -4c_1 e^{-4t} - 2c_2 e^{-2t} \]
\[ y(0) = 3 \implies c_1 + c_2 = 3 \]
\[ y'(0) = 1 \implies -4c_1 - 2c_2 = 1 \]
\[ -2c_1 = 7 \implies c_1 = -\frac{7}{2} \implies c_2 = \frac{13}{2} \]
\[ y = -\frac{7}{2} e^{-4t} + \frac{13}{2} e^{-2t} \]

(c) (10 points) Graph the solution you obtain in part (b), showing the
behavior as \( t = 0 \) and as \( t \to \infty \).
4. The two linearly independent solutions of \( y'' + 25y = 0 \) are \( \sin 5t \) and \( \cos 5t \).

(a) (10 points) Write down the variation of parameter formula for a solution of \( y'' + 25y = e^{t^2} \), evaluating the Wronskian in this formula.

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = \sin 5t \cdot (-5 \sin 5t) - \cos 5t \cdot (5 \cos 5t) = -5
\]

\[
y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} \, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} \, ds
\]

\[
= -\sin 5t \int_{t_0}^{t} \frac{\cos 5s e^{s^2}}{-5} \, ds + \cos 5t \int_{t_0}^{t} \frac{\sin 5s e^{s^2}}{-5} \, ds
\]

(b) (10 points) Use your variation of parameters formula to write down the general solution of \( y'' + 25y = e^{t^2} \).

\[
y(t) + c_1 y_1 + c_2 y_2
\]

\[
= -\sin 5t \int_{t_0}^{t} \frac{\cos 5s e^{s^2}}{-5} \, ds + \cos 5t \int_{t_0}^{t} \frac{\sin 5s e^{s^2}}{-5} \, ds
\]

\[+ c_1 \sin 5t + c_2 \cos 5t\]