Solving Inverse Problems with Spectral Data

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Abstract

We consider a two dimensional membrane. The goal is to find properties of the membrane or properties of a force on the membrane. The data is natural frequencies or mode shape measurements. As a result, the functional relationship between the data and the solution of our inverse problem is both indirect and nonlinear. In this paper we describe three distinct approaches to this problem. In the first approach the data is mode shape level sets and frequencies. Here formulas for approximate solutions are given based on perturbation results. In the second approach the data is frequencies and boundary mode shape measurements; uniqueness results are obtained using the boundary control method. In the third approach the data is frequencies for four boundary value problems. Local existence, uniqueness results are established together with numerical results for approximate solutions.

Introduction

We consider two dimensional membranes. Spectral data is measured for these membranes; that is natural frequencies and/or some specific measurements of the corresponding mode shapes. Choices for the mode shape measurements are level sets or nodal sets in the interior of the membrane or flux or displacement measurements on the boundary. We ask: What can we learn about the membrane from these measurements?

To obtain this data we excite the membrane at a sequence of natural frequencies, often driving the membrane with a time harmonic force at a single point. What results is a wave that travels across the membrane reflecting from the boundary, traveling back, reflecting again and so on. At most frequencies the initial and reflected waves interfere with each other producing a small response. At a natural frequency the initial and reflected waves reinforce each other producing a large response. This response, a combination of traveling waves, makes the membrane appear to be oscillating up and down, we call what we see "a standing wave," and the shape at any instant of time is called a mode shape. The shape is the same at all instances except for a multiplicative or amplitude factor.

When, for example, the edge of the membrane is fixed we can measure level sets of the mode shape by illuminating the membrane with two lasers. The interference pattern is a set of dark and light lines called a holographic image. Each line is a level set. Of course one of these level sets is the nodal set, the set of points that don't vibrate when the membrane is excited at a natural frequency. If, however, we want only the nodal set, a Doppler shift experiment may be considered. There, a single laser is used. As it scans the membrane the Doppler shift in the backscatter is measured and minimum Doppler shift is achieved at nodal points. See [McL2] for more discussion about this.

If, for example, the edge of the membrane is free and our data is displacement of the mode shape at the boundary points together with the corresponding natural frequencies a third experiment can be considered. We can impart an impulse at a point on the membrane. Displacement as a response of the impulse is measured at points around the boundary. We Fourier analyze the response at each point and obtain the natural frequencies and the corresponding displacements at the boundary.

To put these two approaches in perspective we recall what is known in one dimension for Sturm-Liouville problems with separable boundary conditions. The classic approach was initially successfully attacked by Borg [B], Gelfand-Levitan [GL], Hochstadt [Ho] and many others for the differential equation $y'' + (\lambda - q)y = 0$ on $0 \leq x \leq 1$ with separable boundary conditions and $q \in L^2(0,1)$, with a complete solution to the inverse spectral problem for this equation given by [PT], [IT], [IMT]. These results for the inverse problem require two sets of spectral data. In one approach to the problem the data set consists of two sequences of frequencies each for a different set of separable boundary conditions; another data set is one set of frequencies together with mode shape measurements on the boundary (at $x = 0$ and 1), that are different from the boundary conditions. Each data set produces a unique solution; in the work of Trubowitz, et. al., necessary and sufficient conditions that the data determines q together with the boundary conditions are established. An important extension of these results is given in [CMcL1], [CMcL2] where necessary and sufficient conditions, together with formulas for exact solutions, for the inverse spectral problem for $(py')' + \lambda py = 0$, $p > 0$, $p \in H^1(0,1)$ and with Dirichlet boundary conditions are established. This equation cannot be transformed to the one given above with q. For that transformation, $p \in H^2(0,1)$ is needed. Note that the weakening of the smoothness properties of p are important for applications. In fact for applications requiring p to be of bounded variation, i.e. $p \in BV[0,1]$, is often the more realistic assumption, especially when the material properties are expected to have discontinuities. A complete characterization as was established by [PT], [IT], [IMT], for q, and extended by Coleman and McLaughlin [CMcL1], [CMcL2], for $p \in H^1(0,1)$, is still an open problem for $p \in BV[0,1]$. Note, however, that inroads for the $p \in BV[0,1]$ problem have been made by [C], [Ha1], [W1], [W2]. In [Ha2], [Ha3], the author presents an implementation of numerical methods intended for geophysical applications. See also [HMcL4] for new results on the asymptotic behavior of eigenvalues in the BV case.

In a second approach for solving the one dimensional inverse spectral problem, the data is natural frequencies and level set measurements for the corresponding mode shapes. Sufficient conditions for a unique $q \in L^2(0,1)$ are given in [McL1]. Sufficient conditions for a unique $p, \rho \in H^2(0,1)$, for $(py')' + \lambda \rho y = 0$ with separable boundary conditions or an even smoother pair are given in [HMcL2], [HMcL3]. In [HMcL3] numerical implementation of algorithms, that calculate an approximation to one coefficient when the others are known, and that require a natural frequency together with level set measurements from a single mode shape, are given. Error bounds for the difference between the computed approximation and the true solution are established. For these computations, the zero level set, or nodal set, is the one that is used. This work is extended to $p, \rho \in BV[0,L]$ in [HMcL4] (with a significant advance using number theory on the asymptotic behavior of the frequencies). Note that in all cases considered in [HMcL2], [HMcL3], [HMcL4] the algorithms produce piecewise constant approximates; each constant is the difference or ratio of the squares of two frequencies. One of the frequencies is measured and the other is calculated by a very simple formula using only adjacent zero level set data, or adjacent nodes.

This second one-dimensional approach using the zero level set data is extended by [ST], [S], [LY], [LSY]. There new algorithms for q are given and the smoothness of q is established from the actual positions of the zero level set data.

The mathematics for solving two dimensional inverse spectral problems has developed along a number of fronts. Some of the resultant theorems are remarkably similar to the one dimensional results; the techniques however are new. In all cases new results for the direct problem are required. In one case, described in the first section of this paper, perturbation methods are used. Here for the direct problem, perturbation expansions for almost all natural frequencies and mode shapes are developed. In the work completed to date by McLaughlin, Hald, Lee, and Portnoy, [HMcL1], [LMcL], [McLP], [McL3], a rectangular membrane is considered. The goal is to find the (nonconstant) density or a (nonconstant) coefficient in a restoring force. The problem is difficult because even when there is no restoring force and the density is constant the spacing of the eigenvalues, the squares of the natural frequencies, is very irregular, even when all the frequencies are distinct. Some eigenvalues are well-spaced relative to their neighbors while others are clustered together. Perturbation expansions are established when, in the no restoring force - constant density case at least two conditions are satisfied:

1. There is a bound, which decreases as the frequency increases, on the distance to the adjacent frequencies;
2. The distance to selected frequencies is large and increasing as the frequency increases. The selected frequencies have corresponding mode shapes with similar oscillation properties.

Number theory and analysis are used to establish these results. Further, a^2, the square of the ratio of the sides has favorable number theoretic properties; it is irrational and satisfies a Diophantine condition. This aids us to establish the spacing properties above and allows us to write down specific a^2, e.g. $\sqrt{2}$ or $\sqrt{5}$, where our analysis holds.

The hard work to obtain the perturbation expansions pays off. It is applied to the case where the data is a natural frequency and level set measurements of the mode shape. With this data, simple formulas are obtained for the coefficient, q, in the restoring force. In one formula the value of q at selected points is approximated by the difference of two eigenvalues; one is measured, the other is calculated from the data near the selected point. This formula generalizes to two dimensions a corresponding one-dimensional formula. In another formula, q is approximated at selected points by four data points chosen near the selected point. There is no one-dimensional counterpart of this formula, see [HMcL3], [McL3], [McL4]. For each formula error estimates for the difference between q and its approximate value are given.

Even for this data set, which is the eigenfrequency plus the level set measurements, it's possible to eliminate the extensive mathematics needed for the perturbation results if the error bounds are given in terms of additional displacement measurements. The implementation of such an algorithm is given in [LMcL]. The algorithm produces a piecewise constant approximation to the density ρ. Similar to the one-dimensional problem, each constant is the ratio of two frequencies, one that is measured and one that is calculated from zero level set data measured in the neighborhood where that constant approximates ρ.

Note that in all of these solutions the formulas yield local information; i.e. if the solution is required in only part of the membrane, level set data need only be measured in the same region. The convergence of the piecewise constant approximates yields the uniqueness result for q when the domain is a rectangle.

Fewer results have been established when the data, natural frequencies and level set measurements, are given for domains other than rectangular membranes. In [G], however, it is shown that in the q identically equal to zero case, when the domain is a circular disk, if all the nodal lines for all natural frequencies are the same as for the ρ equal constant case, then ρ can only be a constant.

We turn now to other data sets for the two dimensional inverse spectral problem. Both kinds of data sets are suggested by analogous one dimensional inverse problems. For one set of results the data is the eigenfrequencies plus boundary measurements. The boundary measurements, together with the boundary condition, yield full Cauchy data on the boundary of the domain for each eigenfunction. Uniqueness results, see [NSU], [KK], are known and in one case error bounds have been established, see [AS], when only partial data or noisy data is given. Two very different approaches are applied to achieve results. In one approach the spectral data is shown to be sufficient to define the Dirichlet to Neumann, DtN, map. Then properties of the DtN maps are used to establish the uniqueness and error bound results, see [NSU] and [AS]. In the second approach, the model for the membrane is much more general. The proofs do not depend on perturbation results but use the boundary control method of Belishev [Be1], [Be2]. The allowable models are divided into classes defined by groups of gauge transformations. A single element in each class of models is determined by the spectral data. This nonuniqueness when the model class is more general is also seen for one-dimensional problems. These results are described in the second section of this paper. Note that, so far, no numerical algorithms and numerical computations, using this two dimensional data set, have been presented in the literature.

The final inverse spectral problem discussed in this paper uses only eigenvalues as data for the inverse problem. The conjecture is that if there is a single unknown coefficient in the membrane model then the eigenvalues for four eigenvalue problems, each with distinct but related boundary conditions, are enough to determine the unknown coefficient. The last section of this paper describes existence, uniqueness results when the unknown coefficient is close to a constant, is in a finite dimensional space, the domain is a rectangle, and the data is a finite number of eigenvalues. Results from numerical reconstruction in two separate cases (one for q and one for ρ) using this data, are given. This discussion is presented in the last section of this paper.

The inverse spectral problem-using frequency and level set data

We concentrate here on two dimensional results. Under the assumption that the motion of the vibrating membrane is time harmonic we concentrate on the resultant elliptic equation with Dirichlet boundary conditions

$$-T\triangle u + qu = \lambda\rho u, \qquad \underset{\sim}{x} \in R, \tag{1}$$

$$u = 0, \qquad \underset{\sim}{x} \in \partial R. \tag{2}$$

where $R = [0, \pi/a] \times [0, \pi]$ and the eigenvalue $\lambda = \omega^2$ were ω is the natural frequency. Here T is the (constant) tension, q is the (often nonconstant) amplitude of a restoring force and ρ is the (often nonconstant) density per unit area. The two inverse problems we will consider in this section are: (1) recover q/T when ρ/T is a known constant; or (2) recover ρ/T when q/T is identically zero. Note that in one dimension, with enough smoothness, using the Liouville change of variables (see, [BR]) these problems can be transformed one to the other. In two dimensions this is not the case.

We begin with the case where q/T is identically zero. Here we will give a numerical algorithm and demonstrate that it gives rather good results. The method for calculating a piecewise constant approximate for ρ/T when q/T is identically zero when the data set is the zero level set of a mode shape together with the corresponding natural frequency is as follows. Using a graphical display we let Figure 1, 2, and 3 be the nodal set when ρ/T is a constant, the nodal set when ρ/T is not constant, and the division of the nonconstant ρ/T nodal domains into the same number of subdomains, Ω'_j, as in the ρ/T equal constant case.

Figure 1 Figure 2 Figure 3

Note that Figure 2 shows a typical pattern when ρ/T is not constant. Some atypical zero level set patterns occasionally also occur. The method below is not used for these atypical patterns. Further for typical zero level set patterns the diagonal cuts of Figure 3 are known a priori. The piecewise constant approximate is, see [LMcL],

$$(\rho/T)_a = \left\{ \frac{\lambda_{10}(\Omega'_j)}{\lambda_n}, \qquad \underset{\sim}{x} \in \Omega'_j, \right.$$

where $\lambda_n = (\omega_n)^2$ is the square of the natural frequency for the displayed mode shape in Figures 2 and 3 and $\lambda_{10}(\Omega'_j)$ is the smallest eigenvalue for

$$\begin{aligned} -\triangle u &= \lambda u, \qquad && \underset{\sim}{x} \in \Omega'_j, \\ u &= 0, && \underset{\sim}{x} \in \partial\Omega'_j. \end{aligned}$$

Figures 4 and 5 show a nonconstant ρ/T and one calculated approximate $(\rho/T)_a$ calculated from the data from one of the mode shapes.

Figure 4 Figure 5

Note that for this example $R = [0,1] \times [0, e/2]$ and the eigenvalues and nodal position synthetic data are calculated using a spectral method where

$$\rho(x,y) = \begin{cases} 1 + \exp\left(\frac{(0.125)^2 \, ln \, 0.125}{(0.125)^2 - (x-0.75)^2 - (y-1)^2}\right) & \text{for } (x-0.75)^2 + \\ & (y-1)^2 \leq (0.125)^2, \\ 1 & \text{otherwise.} \end{cases}$$

The lowest eigenvalue $\lambda_{10}(\Omega_j')$ for the cut subdomain Ω_j' is calculated using a finite element method. See [Lee] for additional details. Note also that in [Lee] error bounds for the difference between the true $\rho(x,y)$ and its approximate are given. These bounds depend on measurements of the mode shape along the cuts.

Changing now to the problem where ρ/T is constant and q/T is nonconstant we start with the development of perturbation results for the eigenvalues, λ, and the corresponding mode shapes (eigenfunctions). These results are valid for almost all eigenfunctions, including arbitrarily large ones. This job, which is more or less straight forward in one dimension, is made more difficult by the fact that the eigenvalues in two dimensions, which are real, are on average equally spaced but in actual fact are quite irregularly spaced on the real line. Further under sufficient smoothness assumptions on q/T we can show that the biggest change in the mode shape, and hence change in the mode shape level sets, due to a change from constant q/T to nonconstant q/T is made from an interaction with other mode shapes with similar oscillation properties. In addition the eigenvalues corresponding to those mode shapes with similar oscillation properties are not the nearest neighbors on the number line.

To make this more clear we begin now to state some of the hypotheses. The first goal will be to establish results for the spacing of the eigenvalues when q/T is identically zero and ρ/T is a constant where we absorb the constant into the eigenvalues creating a normalization of ρ/T, that is $\rho/T \equiv 1$. Further from now on we'll simply relabel q/T as q. Second we make a choice for a^2; it will be irrational so that all eigenvalues are distinct and satisfy a Diophantine condition. The latter hypothesis allows the utilization of number theoretic arguments in the proofs, considerably shortening the arguments and further allows, because of a fundamental result of Roth [R], that specific a^2, in particular algebraic numbers, can be given for which the theory holds. Specifically the condition required is

The Diophantine Condition:
Let $J = (1, a_0)$ *and* $0 < \epsilon_0 < 2$ *be given. Let* Z *be the set of integers and define*

$$V = \{a \in J \mid \textit{ there exists } 0 < \delta < \epsilon_0/6 \textit{ and } K > 0 \textit{ such that for all } p, q \in Z \\ \textit{with } q > 0 : \mid a^2 - p/q \mid > K/q^{2+\delta}\} .$$

Then V *is of full measure in* J.
Examples of numbers that don't satisfy this condition are given in [HMcL1].

With this assumption the eigenvalue problem

$$-\triangle u = \lambda u, \qquad \underset{\sim}{x} \in R, \tag{3}$$
$$u = 0, \qquad \underset{\sim}{x} \in \partial R, \tag{4}$$

is considered; it has eigenvalue, (normalized) eigenfunction pairs

$$\lambda_{\alpha 0} = a^2 n^2 + m^2,$$
$$u_{\alpha 0} = \frac{2\sqrt{a}}{\pi} \sin anx \sin my,$$

for each $\alpha = (an, m)$, n, m positive integers. Notice that α is an element in a lattice plane $L = \{\alpha = (an, m) \mid \ n, m > 0, n, m \in Z\}$ while $\lambda_{\alpha 0} = \mid \alpha \mid^2$ is a point on the real line. Requiring spacing properties for $\lambda_{\alpha 0}$ on the number line, sometimes relative to the position of the corresponding α in the lattice plane, is an essential part in the perturbation expansion. Specifically it is established that

Lemma 1:
For $0 < \delta < \epsilon_0/6$, *and* a^2 *satisfying the Diophantine condition, that is* $a^2 \in V$, *the set*

$$M_{10}(a) = \{\alpha \in L \mid \textit{ there exists } \beta \in L, \ \beta \neq \alpha, \mid\mid \alpha \mid^2 \ - \ \mid \beta \mid^2 \mid < 4 \mid \alpha \mid^{-\epsilon_0}\}$$

satisfies

$$\lim_{r \to 0} \frac{\# M_{10}(a) \cap \{\alpha \in L \mid \ \mid \alpha \mid < r\}}{\# \{\alpha \in L \mid \ \mid \alpha \mid < r\}} = 0.$$

That is $M_{10}(a)$ has density zero in L.

Note that this lemma says that for almost all $\alpha \in L$ there is a lower bound, that slowly decreases as $\mid \alpha \mid$ gets large, on the distance from $\lambda_{\alpha 0} = \mid \alpha \mid^2$ to the nearest neighbor, $\lambda_{\beta 0} = \mid \beta \mid^2$. This is illustrated in the graph in Figure 6.

Figure 6

The second spacing lemma is:

Lemma 2:
For $0 < \epsilon_1 + 5\delta + \epsilon_1 \delta < \epsilon_2 < \frac{1}{2}$, $C_1, C_2 \geq 1$ and $a^2 \in V$, the set

$$\begin{aligned} M_{11}(a) = \{\alpha \in L \mid & \textit{ there exists } \ \beta \in L, \textit{ with } \beta \neq \alpha, \textit{ and} \\ & \mid \alpha - \beta \mid < C_1 \mid \alpha \mid^{\epsilon_1}, \\ & \mid \lambda_{\alpha 0} - \lambda_{\beta 0} \mid < C_2 \mid \alpha \mid^{1-\epsilon_2} \} \end{aligned}$$

has density zero in L, that is

$$\lim_{r \to \infty} \frac{\# \{\alpha \in M_{11}(a) \mid \ \mid \alpha \mid < r\}}{\# \{\alpha \in L \mid \ \mid \alpha \mid < r\}} = 0.$$

This lemma establishes the fact that for almost all α, the lattice points β, that are near α in the lattice plane, and so have the property that the corresponding mode shapes $u_{\alpha 0}, u_{\beta 0}$ have similar oscillation properties, also have the property that $\lambda_{\alpha 0} = |\alpha|^2$ and $\lambda_{\beta 0} = |\beta|^2$ are a large distance apart on the number line. A graphical illustration of this is in Figure 7.

Figure 7

For this lemma we comment that it is slightly different than the corresponding Lemma 1.2 in [HMcL1]. The proof for the above result is shorter, using number theoretic arguments, while the hypothesis on ϵ_1 and ϵ_2 is slightly stronger.

One additional condition is needed to establish the perturbation results. In [HMcL1], since an inverse problem is solved, it is assumed that the location of the eigenvalues $\{\lambda_{jq}\}_{j=1}^{\infty}$ for

$$-\triangle u + qu = \lambda u \qquad \underset{\sim}{x} \in R, \tag{5}$$

$$u = 0, \qquad \underset{\sim}{x} \in \partial R, \tag{6}$$

are known. Then, without loss of generality, it is assumed that $\int_R q = 0$. The third condition becomes

Lemma 3:
Let

$$L \backslash M = \{\alpha \in L \backslash M_{10} \cup M_{11} \mid \textit{ exactly one } \lambda_{kq} \textit{ is in the interval } (\mid \alpha \mid^2 - 2 \mid \alpha \mid^{-\epsilon_0}, \mid \alpha \mid^2 + 2 \mid \alpha \mid^{-\epsilon_0}\} .$$

Then M has density zero, that is

$$\lim_{r \to \infty} \frac{\{\alpha \in M \mid \ \mid \alpha \mid < r\}}{\{\alpha \in L \mid \ \mid \alpha \mid < r\}} = 0.$$

While Lemma 3 is satisfactory for the inverse problem, it would be stronger if the condition did not depend on q. Indeed in [McLP], this improvement is accomplished. The alternate lemma is:

Lemma 3A:
Let $0 < \epsilon_0 < (\epsilon_2 - \epsilon_1)/2,\ 0 < \epsilon_3 < \epsilon_2 - \epsilon_1 - 2\epsilon_0$. Let $a \in V$. Define

$$M_{13}(a) = \{\alpha \in L\backslash M_{10} \mid \textit{there exists } \beta, \gamma \in L,\ \beta, \gamma \neq \alpha, \\ 0 < \mid \beta - \gamma \mid < C_1 \mid \alpha \mid^{\epsilon_1}, \\ \mid \lambda_{\beta 0} - \lambda_{\alpha 0} \mid \mid \lambda_{\gamma 0} - \lambda_{\alpha 0} \mid < C_3 \mid \alpha \mid^{\epsilon_3} \}\,.$$

Then M_{13} has density zero. Further if $\breve{M} = M_{10} \cup M_{11} \cup M_{13}$ then $\breve{M}$ is of density zero in L.

Note that when $\alpha \in L\backslash M_{10} \cup M_{11}, \cup M_{13}$, then it is proved that $\alpha \in L\backslash M$.

Finally the smoothness condition for q is given in terms of

$$\mid q \mid_\ell = \left\{ \sum_{\alpha \in L'} \mid \alpha \mid^{2\ell} \mid a_\alpha \mid^2 \right\}^{1/2}$$

where $L' = \{\alpha = (an, m) \mid n, m \geq 0, n, m \in Z\}$, where a_α are the Fourier coefficients for q for the basis $v_\alpha = c_\alpha \cos{(anx)} \cos{my},\ \alpha \in L'$, and where

$$c_\alpha = \begin{cases} \frac{2\sqrt{a}}{\pi} \ \ if \quad n \neq 0, \ \ m \neq 0 \\ \frac{\sqrt{2a}}{\pi} \ \ if \ \ n = 0, \ \ m \neq 0 \ \ or \ \ n \neq 0, \ m = 0 \\ \frac{\sqrt{a}}{\pi} \ \ if \quad n = 0, \quad m = 0 \end{cases}$$

The precise conditions on ℓ, C_1, C_2 and C_3 are given in [HMcL1], [McLP], as well as the perturbation expansions. Note that for each $\alpha \in L\backslash \breve{M}$, or $\alpha \in L\backslash M$, there is a unique eigenvalue, eigenfunction pair $\{\lambda_{\alpha q}, u_{\alpha q}\}$ that is a perturbation of $\{\lambda_{\alpha 0}, u_{\alpha 0}\}$. Further, in [HMcL1] the exact error estimates needed for the inverse problem solution are derived while in [McLP] a full perturbation series is established.

Turning now to the solution of the inverse problem, formulas are given in the following three theorems. To frame the first result we show graphically how we subdivide the membrane. When $\alpha \in L\backslash M$ (or $\alpha \in L\backslash \breve{M}$) satisfies our conditions, the zero level set for $u_{\alpha 0}$ and the zero level set for $u_{\alpha q}$, when q is not identically zero, are shown in Figures 8 and 9; in Figure 10 we show the subdivision of the domain into nm subregions Ω'_j using the zero level set in Figure 9 and diagonal straight line cuts. The straight lines for the diagonal cuts are known apriori and are described in [HMcL1]. Notice the similarity with the Figures 1, 2 and 3.

Figure 8 Figure 9 Figure 10

Then we form the piecewise constant approximate q_a

$$q_a = \left\{ \lambda_{\alpha q} - \lambda_{10}(\Omega_j') \qquad \underset{\sim}{x} \in \Omega_j',\ j = 1, ..., nm \right.$$

where $\lambda_{10}(\Omega_j')$ is the smallest eigenvalue for

$$\begin{aligned} -\triangle u &= \lambda u, \qquad & \underset{\sim}{x} \in \Omega_j', \\ u &= 0, \qquad & \underset{\sim}{x} \in \partial\Omega_j', \end{aligned}$$

$j = 1, 2, ..., m$. Then it is proved that,

Theorem 1:
Let $\alpha \in L \backslash M$ (or $\alpha \in L \backslash \breve{M}$). Then in each Ω_j', $j = 1, 2, ..., nm$ there exists an x_j' with

$$\mid q(x_j') - q_\alpha(x_j') \mid < \frac{1}{9 \mid \alpha \mid^{2-4\alpha_2}}.$$

The error estimate in Theorem 1 is valid even when $\int_R q \neq 0$. Note that similar results have been obtained, see [VA], when the Dirichlet boundary conditions are replaced by mixed boundary conditions.

While this is an elegant formula, extensive numerical computations may be needed to compute each $\lambda_{10}(\Omega_j')$ to obtain $q_\alpha(x_j')$. Surprisingly, we can simplify this a great deal. Again we show the idea graphically. In Figure 11 we repeat

Figure 8 adding the dashed midlines; in Figure 12 we repeat Figure 9 adding the midlines of Figure 11.

Figure 11 Figure 12

Again approximate q by a piecewise constant function but now there is a new formula for the approximate q_{aa}

$$q_{aa} = \left\{ \lambda_{\alpha q} - 3\lambda_{\alpha 0} - \frac{2}{\pi}[(an)^3 \ell_x + m^3 \ell_y], \quad \underset{\sim}{x} \in \Omega_j \ , \right.$$

$j = 1, ..., nm$, where Ω_j is any subdomain in Figure 8 (or 11) and ℓ_x and ℓ_y are chosen for the same subdomain. Note that the calculation here is very straight forward requiring only simple differences of the measured data. To choose the lengths ℓ_x and ℓ_y let x_j'' be the point of intersection of the dashed midlines in Ω_j. Then ℓ_y is the length of the vertical dashed midline passing through x_j'' and measured from the closest nodal point below x_j'' to the closest nodal point above x_j''. The length ℓ_x is the corresponding horizontal distance. We can establish

Theorem 2:

Let $\alpha \in L \backslash M$ (or $\alpha \in L \backslash \breve{M}$). Then there exists a constant C with

$$| \ q \ (x_j'') \ - \ q_{aa}(x_j'') \ | \leq \ C \ | \ \alpha \ |^{-2+(3/2)\epsilon_2} \ .$$

Finally there are several similar formulas we can establish using nonzero level sets. We give only one here again using a graphical representation. Figure 13 shows a non zero level set.

Figure 13

The piecewise constant approximate

$$q_{aaa} = \{ \lambda_{\alpha q} - 3\lambda_{\alpha 0} + 2\left((an)^2 2x^e + m^2 2y^e\right) \\ + \frac{2}{\pi}\left((an)^3 \ell_x^e + m^3 \ell_y^e\right) \qquad \underset{\sim}{x} \in \Omega_j,$$

where x^e and y^e are given apriori and do not depend on the data; error bounds for $q - q_{aaa}$ at x_j'', similar to those in Theorem 2, can be established, see [McL4].

The inverse spectral problem-using boundary, spectral data

In this section we briefly review results for inverse spectral problems where the data is eigenvalues and boundary data for the eigenmodes. The boundary data is chosen so that full Cauchy data is known for the eigenmodes on the entire boundary of the region. Two approaches for establishing results have been used. In one approach, see [NSU], the spectral data is shown to establish the Dirichlet to Neumann (DtN) map and then results for DtN maps are used to achieve results. This is a clever use of existing results. Further an important estimate of the error, see [AS], when only partial spectral data is known and when the data may contain an order ϵ error is established. A second approach yielding an extensive set of results, see e.g. [BK] and [KK], and relies on the boundary control method first put forth by Belishev, [Be1], for isotropic inverse problems. This work has required the development of significant new mathematics and has been generalized to mathematical models that can include anisotropy.

Note that the full set of data considered here, eigenvalues plus boundary data is more than is needed to achieve a solution to the inverse problem. The richness of this data is evident in the fact that a full matrix of coefficients which could represent anisotropy in a physical medium can be determined by the data, even when a finite number of eigenvalues and the boundary eigenmode data for the corresponding eigenmodes is omitted. Note also, in some cases it is possible to use only partial data on the boundary. We do not give these results here but refer the reader to [Be2, p. R39] for a discussion.

Here we restrict the statement of results to two dimensions even though the original statements of the theorems are stated for dimension $n \geq 2$. We begin by quoting one of the first results in [NSU].

Theorem 3:
Let Ω be a bounded domain in R^2 with smooth boundary. Let $q_i \in L^\infty(\Omega), i = 1, 2,$ and consider the eigenvalue problems

$$-\Delta u + q_i u = \mu u, \qquad \underset{\sim}{x} \in \Omega,$$

$$u = 0, \qquad \underset{\sim}{x} \in \partial\Omega$$

Denote $\{\mu_j(q_i), \phi_j(x; q_i)\}_{j=1}^\infty$ as the eigenvalue, eigenfunction pairs for the above eigenvalue problems, $i = 1, 2,$ and suppose that

$$\mu_j(q_1) = \mu_j(q_2), \qquad j = 1, 2, \ldots,$$

$$\frac{\partial\phi_j(x; q_1)}{\partial\nu} = \frac{\partial\phi_j(x; q_2)}{\partial\nu}, \qquad j = 1, 2, \ldots, \qquad \underset{\sim}{x} \in \partial\Omega$$

where ν is the unit outward normal to $\partial\Omega$. Then $q_1(x) = q_2(x)$ for all $x \in \Omega$.

Note that although the statement of the above theorem has a Dirichlet boundary condition in the eigenvalue problem, this condition is not necessary for the result; the boundary condition can be changed to $\partial u/\partial\nu + \alpha u = 0$ for $\underset{\sim}{x} \in \partial\Omega$ and α a smooth real function in $\partial\Omega$. In this latter case the condition $\partial\phi_j(x; q_1)/\partial\nu = \partial\phi_j(x; q_2)/\partial\nu$ for $x \in \partial\Omega$ is changed to $\phi_j(x; q_1) = \phi_j(x; q_2)$ for $x \in \partial\Omega, j = 1, 2, \ldots$ With these changes, the conclusion of the theorem is unchanged.

Uniqueness results suggest that stability results can follow. Such results are established in [AS] where the main theorem is (again stated only for two dimensions)

Theorem 4:
Let Ω be a bounded domain in R^2 with smooth boundary. Let $q_i, i = 1, 2$ be bounded Hölder continuous functions in $\bar{\Omega}$ with

$$\|q_i\|_{L^\infty(\Omega)} \leq M,$$

$$| q_i(\underset{\sim}{x}) - q_i(\underset{\sim}{y}) | \leq E \, | \underset{\sim}{x} - \underset{\sim}{y} |^\alpha, \quad \underset{\sim}{x}, \underset{\sim}{y} \in \bar{\Omega}$$

for some $M, E > 0$ and $0 < \alpha < 1$. Let $\{\lambda_j^i, \phi_j(x; q_i)\}_{j=1}^\infty$, be the eigenvalue, eigenfunction pairs for

$$-\Delta u + q_i u = \mu u, \qquad \underset{\sim}{x} \in \Omega,$$

$$u = 0, \qquad \underset{\sim}{x} \in \partial\Omega,$$

$i = 1, 2$. *Then there exist positive constants* A, B, C *and* $0 < \sigma < 1$ *such that for every* $N > 0$,

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C(N^A \epsilon^\sigma + N^{-B}),$$

where

$$\epsilon = \sup_{j \leq N} \mid \lambda_j^1 - \lambda_j^2 \mid + \sup_{j \leq N} \left\| \frac{\partial \phi_j(x; q_1)}{\partial \nu} - \frac{\partial \phi_j(x; q_2)}{\partial \nu} \right\|_{L^\infty(\partial\Omega)}.$$

The authors point out that if the eigenfunctions are not chosen carefully in the multiple eigenvalue case then it is possible to have $\epsilon > 0$ even when $q_1 \equiv q_2$. Note also that the constants A, B and σ in the theorem depend only on the Hölder constant α and the space dimension. The proof of this result exploits the connection with the DtN map.

We turn now to another set of results where the mathematics that provides the proofs of these results is based in differential geometry. Instead, then, of speaking about anisotropic media, the authors of these results speak about a compact, connected, oriented differentiable (C^∞) manifold, M, with dim $M = 2$ (again we restrict our statements to two dimensions) and smooth non-zero (one) dimensional manifold boundary $S = \partial M$. The Riemannian metric is denoted by g on M with associated measure $dV = dV_g$. What is considered then is the operator

$$\mathcal{A}u = a(x, D)u = -g^{1/2} \left(\frac{\partial}{\partial x_k} + ib_k \right) \left(g^{1/2} g^{k\ell} \mu \left(\frac{\partial}{\partial x_\ell} + ib_\ell \right) u \right) + qu, \quad (7)$$

where a sum over k and ℓ is understood implicitly. Note that the metric tensor $(g^{k\ell})_{k,\ell,=1,2}$ is symmetric,

$$g(x) = \{\det[g^{k\ell}]\}^{-1}, \quad dV_g = g^{1/2} dx_1 dx_2,$$

and $\mu > 0$, with $q, b_j, j = 1, 2$, being real valued and C^∞ smooth, and $b_j, j = 1, 2$ forming a differential 1 - form on M. The above operator is considered together with the boundary condition

$$\mathcal{B}u = \left(\frac{\partial}{\partial \nu} + ib \cdot \nu + \sigma \right) u = 0, \qquad \underset{\sim}{x} \in S, \quad (8)$$

where σ is a real valued smooth function defined on S.

The goal of the inverse problem is to recover the coefficients in $\mathcal{A}$ and $\mathcal{B}$ from incomplete boundary spectral data (IBSD) defined as follows:

Definition:

Let N be the set of positive integers and $K' \subset N$ be a finite subset. Let $\{\lambda_k, \phi_k\}_{k\in N}$ be the eigenvalue, eigenfunction pairs for

$$\begin{aligned} \mathcal{A}u = \lambda u, \qquad & \text{on } M, \\ \mathcal{B}u = 0 \qquad & \text{on } S. \end{aligned} \tag{9}$$

Then the collection $(S, \{\lambda_k\}_{k\in N-K'}, \{\phi_k \mid_S\}_{k\in N-K'})$ is called the incomplete boundary spectral data (IBSD) for the operator $\mathcal{A}$ (together with $\mathcal{B}$).

Optimistically one might expect that the IBSD would determine all the coefficients in $\mathcal{A}$ and $\mathcal{B}$ but such is not the case. It happens that there is a set of transformations that leave the eigenvalues fixed, that multiply each eigenfunction on the boundary by the same function, that leave the manifold unchanged but change $\mathcal{A}$ and $\mathcal{B}$; this is the group $\mathcal{G}$ of generalized gauge transformations. In any orbit of $\mathcal{G}$, see[KK], there is a unique canonical representation called the Schrödinger operator (with magnetic potential) (see [KK]). The uniqueness result that can then be obtained is (without specifically defining the canonical Schrödinger operator explicitly) is

Theorem 5:
Let $\mathcal{A}$ together with $\mathcal{B}$ be a canonical Schrödinger operator. Then its IBSD $(S, \{\lambda_k\}_{k\in N-K'}, \{\phi_k \mid_s\}_{k\in N-K'})$ determines $\mathcal{A}$ and $\mathcal{B}$, i.e. $(M, g), q, \sigma$, and b uniquely.

The proof, which is quite extensive, does not rely on perturbation methods. Rather detailed results about Gaussian beams, which are rapidly oscillating solutions of $\mathcal{A}u(x,t) + \frac{\partial^2}{\partial t^2}u(x,t) = 0$, concentrated near a space-time ray, are needed to recover coefficients, such as q, which have a low order effect on the spectral data.

Note that the need to restrict to a canonical problem in order to achieve uniqueness is mirrored in one dimension. There the situation is this. We consider the eigenvalue problem

$$\begin{aligned} (pu_x) - q + \lambda\rho u = 0, \qquad & 0 < x < 1, \\ (u_x + au) \mid_{x=0} = 0, \quad (u_x + bu) \mid_{x=1} = 0, & \end{aligned} \tag{10}$$

where $p_{xx}, \rho_{xx}, q \in L^2(0,1), p, \rho > 0$. Then multiply λ by a constant and divide ρ by the same constant so that the resultant p, ρ satisfy $\int_0^1 \sqrt{\rho/p}dx = 1$. Following that make the change of dependent and independent variables (the Liouville transformation)

$$\begin{aligned} s &= \int_0^x \sqrt{\rho/p}dx, \\ v &= (p\rho)^{1/4}u, \end{aligned}$$

to obtain that v satisfies the equations

$$\begin{aligned} v_{ss} - ((q/\rho) + [((p\rho)^{1/4})_{ss}/(p\rho)^{1/4}])v + \lambda v = 0, \qquad & 0 < s < 1, \\ (v_s\sqrt{\rho/p} + av) \mid_{s=0}, \ (v_s\sqrt{\rho/p} + bv) \mid_{s=1} = 0. & \end{aligned} \tag{11}$$

The boundary spectral data becomes

$$\{\lambda_j, v_j(0), v_j(1)\}_{j=1}^{\infty}$$

where$\{\lambda_j, v_j\}_{j=1}^{\infty}$ are the eigenvalue, eigenfunction pairs for (11). Since each v_j can be multiplied by a constant, the information content in $v_j(0)$ and $v_j(1)$ is contained in $v_j(1)/v_j(0)$. It is known see e.g. [IT], [IMT] that the data

$$\{\lambda_j, v_j(1)/v_j(0)\}_{j=1}^{\infty}$$

is exactly the right amount of data to uniquely determine the triple

$$\left\{\begin{array}{c} \hat{a} \\ \\ \hat{b} \\ \\ \hat{q}(s) \end{array}\right\} = \left\{\begin{array}{l} a/\sqrt{\rho/p(0)} \\ \\ b/\sqrt{\rho/p(1)} \\ \\ (q/\rho) + [((p\rho)^{1/4})_{ss}/(p\rho)^{1/4}] \end{array}\right\} \tag{12}$$

It is not possible, however, to recover the three functions q, p, ρ from (12) uniquely. In fact there is a whole class of Liouville transformations that could be applied to obtain problems of the form (10) from the given data. Three possibilities include:

(I) $p, \rho \equiv 1$, $s = x$, $a = \hat{a}$, $b = \hat{b}$, $q = \hat{q}$;

(II) $q = c\rho$ with $c < \min_j \lambda_j$, $p \equiv \rho$, $s = x$, $a = \hat{a}$, $b = \hat{b}$, and p is a positive solution of $(p^{1/2})_{ss} - (\hat{q} - c)p^{1/2} = 0$;

(III) $q \equiv c < \min_j \lambda_j$, $\rho \equiv 1$, with p a positive solution of $(p^{1/4})_{ss} - (\hat{q} - c)p^{1/4} = 0$, satisfying $\int_0^1 \sqrt{p(s)}ds = 1$ and the original independent variable $x = \int_0^s \sqrt{p(s)}ds$, also $a = \hat{a}\sqrt{p(0)}$, $b = \hat{b}\sqrt{p(1)}$.

The inverse spectral problem-using only eigenvalues

It is important to include one more set of results and these results address the problem where one recovers material properties using only eigenvalues. To address this challenging problem we first recall that, in one dimension, if we know the eigenvalues for the following two problems, with the same $q \in L^2(0, 1/2)$,

$$\begin{aligned} y'' + (\lambda - q)y = 0 \qquad 0 < x < \frac{1}{2} \\ y(0) = y(\frac{1}{2}) = 0 \end{aligned} \tag{13}$$

with eigenvalues $\lambda_1, \lambda_2, ...$ and

$$z'' + (\mu - q)z = 0 \qquad 0 < x < \frac{1}{2} \tag{14}$$
$$z(0) = z'(\frac{1}{2}) = 0$$

with eigenvalues $\mu_1, \mu_2, ...$

then there is at most one $q \in L^2(0, 1/2)$ with these sets of eigenvalues. This can be established in a straight forward manner from the results in [PT] and is addressed in [L]. It is also equivalent to the following. First extend q to q_e, defined on $0 < x < 1$, by making an even reflection of q about $x = 1/2$. Then the combined set $\{\lambda_i\}_{i=1}^{\infty} \cup \{\mu_i\}_{i=1}^{\infty}$ is the set of eigenvalues for

$$w'' + (\eta - q_e)w = 0, \qquad 0 < x < 1, \tag{15}$$
$$w(0) = w(1) = 0.$$

An independent proof for (15), see [PT], shows that there is at most one symmetric $q_e \in L^2(0,1)$ for which the combined set $\{\lambda_i\}_{i=1}^{\infty}, \cup \{\mu_i\}_{i=1}^{\infty}$ are the eigenvalues of (15).

These one dimensional results suggest two related two dimensional inverse spectral problems. The basic idea is this. If in one dimension two sets of eigenvalues provide a uniqueness result for a nonsymmetric q then are four sets of eigenvalues enough to provide a uniqueness result in two dimensions? To illustrate, consider the following example. Let $\hat{R}$ be the rectangle $\hat{R} = [0, \pi/2a] \times [0, \pi/2]$ with $\partial_1 \hat{R} = \{(\pi/2a, y) \mid y \in [0, \pi/2]\}, \partial_2 \hat{R} = \{(x, \pi/2) \mid x \in [0, \pi/2a]\}, \partial_3 \hat{R} = \{(0, y) \mid y \in [0, \pi/2]\}$ and $\partial_4 \hat{R} = \{(x, 0) \mid x \in [0, \pi/2a]\}$ subsets of ∂R. Consider the four eigenvalue problems with the same $q \in L^{\infty}(\hat{R})$ and $\underset{\sim}{n}$ the unit outward normal to points on the $\partial \hat{R}$:

$$\triangle u + (\lambda - q)u = 0, \qquad \underset{\sim}{x} \in \hat{R}, \tag{16}$$
$$u = 0, \qquad \underset{\sim}{x} \in \partial \hat{R},$$

with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$,

$$\triangle u + (\lambda - q)u = 0, \qquad \underset{\sim}{x} \in \hat{R}, \tag{17}$$
$$u = 0, \qquad \underset{\sim}{x} \in \partial_1 \hat{R} \cup \partial_3 \hat{R} \cup \partial_4 \hat{R},$$
$$\triangledown u \cdot \underset{\sim}{n} = u_y = 0 \qquad \underset{\sim}{x} \in \partial_2 \hat{R},$$

with eigenvalues $\mu_1 < \mu_2 \leq \mu_3 \leq ...$,

$$\triangle u + (\nu - q)u = 0 \qquad \underset{\sim}{x} \in \hat{R}, \tag{18}$$
$$u = 0 \qquad \underset{\sim}{x} \in \partial_2 \hat{R} \cup \partial_3 \hat{R} \cup \partial_4 \hat{R},$$
$$\triangledown u \cdot \underset{\sim}{n} = u_x = 0, \qquad \underset{\sim}{x} \in \partial_1 \hat{R},$$

with eigenvalues $\nu_1 < \nu_2 \leq \nu_3 \leq ...$, and

$$\begin{aligned} \triangle u + (\eta - q)u = 0, \qquad & \underset{\sim}{x} \in \hat{R}, \\ u = 0, \qquad & \underset{\sim}{x} \in \partial_3 \hat{R} \cup \partial_4 \hat{R}, \\ \nabla u \cdot \underset{\sim}{n} = 0 \qquad & \underset{\sim}{x} \in \partial_1 \hat{R} \cup \partial_2 \hat{R}, \end{aligned} \tag{19}$$

with eigenvalues $\eta_1 < \eta_2 \leq \eta_3 \leq ...$

By evenly reflecting q about $\partial_1 \hat{R}$ and then evenly reflecting the resultant q about the resultant extension of $\partial_2 \hat{R}$ then we obtain the eigenvalue problem

$$\begin{aligned} \triangle u + (\lambda - q_e)u = 0, \qquad & \underset{\sim}{x} \in R = [0, \pi/a] \times [0, \pi], \\ u = 0, \qquad & \underset{\sim}{x} \in \partial R, \end{aligned} \tag{20}$$

with eigenvalues $\{\lambda_i\}_{i=1}^{\infty} \cup \{\mu_i\}_{i=1}^{\infty} \cup \{\nu_i\}_{i=1}^{\infty} \cup \{\eta_i\}_{i=1}^{\infty}$ and where q_e is the even extension of q to all of R. Hence for this particular problem, solving the inverse problem of finding a symmetric $q_e \in L^{\infty}(R)$ from the eigenvalues of (20) is equivalent to finding a nonsymmetric $q \in L^{\infty}(\hat{R})$ from the eigenvalues of (16), (17), (18), (19).

To illustrate the known results then we consider only (20) and a related problem

$$\begin{aligned} \frac{1}{\lambda} \triangle u + \rho u = 0, \qquad & \underset{\sim}{x} \in R, \\ u = 0, \qquad & \underset{\sim}{x} \in \partial R, \end{aligned} \tag{21}$$

where in this related problem the goal is to solve the inverse problem: find a symmetric $\rho > 0$ from the eigenvalues of (21).

Two local results are known. Both produce functions q or ρ for which (20), or respectively (21), have a given finite number, say m, eigenvalues. Both extended a method first developed by Hald, [Ha1], for the one dimensional inverse spectral problem. The basic idea is to establish an approximate matrix eigenvalue problem for (20) (and similarly (21)) where the approximate problem is derived using spectral approximations. For each matrix problem q (or ρ) is assumed to be in the span of m given basis functions and each eigenfunction is in the span of N (possibly different) basis functions, $N \geq m$. With m fixed for each N a function q_N (or ρ_N is determined as a solution of the corresponding $N \times N$ matrix inverse eigenvalue problem. As $N \to \infty, q_N \to q$ (or $\rho_N \to \rho$) a function in the span of the m basis functions. For that q or ρ the eigenvalue problem (20) (and similarly (21)) has the given finite set of eigenvalues.

Specifically in [KMcL] and [McC], the following are proved. For (20),

Theorem 6:
Let $\left\{\lambda_n^0\right\}_{n=1}^m = \Lambda^0$ *be the first m eigenvalues for*

$$\begin{aligned} \triangle u + \lambda u = 0, \qquad & \underset{\sim}{x} \in R, \\ u = 0, \qquad & \underset{\sim}{x} \in \partial R, \end{aligned}$$

with $R = [0, \pi/a] \times [0, \pi], a > 0$ *and with* $min_{1 \leq n \neq n' \leq m} \mid \lambda_n^0 - \lambda_{n'}^0 \mid = \delta > 0$. *Let* $\Lambda = \{\lambda_n\}_{n=1}^m$ *, all distinct, be given along with a set of symmetric, orthonormal basis functions* $\{\psi_n\}_{n=1}^m$*, each symmetric on R. Then there exists* $\delta_1, \delta_2 > 0, \{\beta_n\}_{n=1}^m$ *and symmetric* $q = \sum_{n=1}^m \beta_n \psi_n$ *with* $\| \Lambda - \Lambda^0 \|_{\ell^2} < \delta_1$, $\| q \|_{L_\infty} < \delta_2$ *and with the property that* $\Lambda = \{\lambda_n\}_{n=1}^m$ *are the first m eigenvalues of*

$$\begin{aligned} \triangle u + (\lambda - q) u = 0, \quad & \underset{\sim}{x} \in R, \\ u = 0, \quad & \underset{\sim}{x} \in \partial R. \end{aligned}$$

And for (21),

Theorem 7:
Let $\left\{1/\lambda_n^0\right\}_{n=1}^m = \Gamma^0$ *be the first m eigenvalues for*

$$\begin{aligned} \frac{1}{\lambda} \triangle u + u = 0, \qquad & \underset{\sim}{x} \in R, \\ u = 0, \qquad & \underset{\sim}{x} \in \partial R. \end{aligned}$$

with $min_{1 \leq n \neq n' \leq m} \mid 1/\lambda_n^0 - 1/\lambda_{n'}^0 \mid = \delta > 0$. *Let* $\Gamma = \{1/\lambda_n\}_{n=1}^m$*, all distinct be given along with a set of symmetric orthonormal basis functions* $\{\psi_n\}_{n=1}^m$*. Then there exist* $\delta_1, \delta_2 > 0, \{\beta_n\}_{n=1}^m$*, and symmetric* $\rho = 1 + \sum_{n=1}^m \beta_n \psi_n$ *with* $\| \Gamma - \Gamma^0 \|_{\ell^2} < \delta_1, \| \rho - 1 \|_{L^\infty} < \delta_2$ *such that* $\Gamma = \{1/\lambda_n\}_{n=1}^m$ *are the first m eigenvalues for*

$$\begin{aligned} \frac{1}{\lambda} \triangle u + \rho u = 0, \qquad & \underset{\sim}{x} \in R, \\ u = 0 \qquad & \underset{\sim}{x} \in \partial R. \end{aligned}$$

Note that in [KMcL] and [McC], the constants δ_1 and δ_2 are given explicitly in terms of δ. Note also that both theorems are proved using contraction mappings and the same idea is used for the numerical algorithms. Examples to show achieved results from the numerical computations are contained in Figures 14, 15, 16 and Figure 17. Figure 14 exhibits

$$q_1(x,y) = \begin{cases} \exp\left(\frac{-1}{d(x,y)}\right) & \text{if } d(x,y) = 1 - (x - \frac{\pi}{2a})^2 - 3(y - \frac{\pi}{2})^2 > 0, \\ 0 & \text{otherwise}, \end{cases}$$

with $a = \sqrt{0.9}$, used to compute the eigenvalues $\{\lambda_n\}_{n=1}^8$. The eigenvalues are calculated using a Matlab finite element package. Figure 15 shows the projection of q_1 onto the span of $\{\psi_n\}_{n=1}^8$ where each

$\psi_n = (2\sqrt{a}/\pi)\sin((2s_n - 1)ax)\sin((2t_n - 1)y)$ with (s_n, t_n) distinct pairs of integers for $n = 1, ..., 8$; Figure 16 exhibits the reconstruction of the approximate $q_1 = \sum_{n=1}^{8} \beta_n \psi_n$ using the matrix approximation of (20), the data $\{\lambda_n\}_{n=1}^{8}$ and the resultant fixed point iteration to solve the matrix inverse problem for $\{\beta_n\}_{n=1}^{8}$ when $N = 64$.

Figure 14

Figure 15

Figure 16

Figure 17 shows the results of the computation for approximate $\rho = 1 + \sum_{n=1}^{10} \beta_n \psi_n$ in (21) from given $\Gamma = \{1/\lambda_n\}_{n=1}^{10}$. Here the eigenvalue data is again calculated with a Matlab finite element tool box and with

$$\rho = \begin{cases} 1 + \exp\left(\frac{-1}{d(x,t)}\right) \text{ if } d(x,y) = \left(\frac{\pi}{3}\right)^2 - 4\left(x - \frac{\pi}{2a}\right)^2 - \left(y - \frac{\pi}{2}\right)^2 > 0, \\ 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

and with $a = \sqrt{0.85}$.

Note that for this problem (21), new theoretical complications arise partly because the effect of changes in ρ on the eigenvalues (and vice-versa) is rather strong.

Figure 17

Determining whether four sets of eigenvalues is enough to determine q (or ρ) for general domains, for q (or ρ) in an infinite dimensional space, and for q (or ρ) not sufficiently close to a constant is an open problem.

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