

# Relative Equilibria of Point Vortices on the Sphere

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September 30, 1999

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## Introduction

Since the work of Helmholtz [12] and Kirchoff [18] systems of point vortices on the plane have been widely studied as finite dimensional approximations to vorticity evolution in fluid dynamics. Small numbers of point vortices model the dynamics of concentrated regions of vorticity while large numbers can be used to approximate less concentrated regions. The equations of motion can be derived by substituting delta functions into the vorticity equation for a two dimensional ideal fluid. For general surveys of planar point vortex systems see for example [2, 3, 4, 7, 39].

The analogous systems on the sphere provide simple models for the dynamics of concentrated regions of vorticity, such as cyclones and hurricanes, in planetary atmospheres. The effects of ‘background’ continuous distributions of vorticity, such as the planetary vorticity of a rotating sphere, can also be included, but are not considered in this paper. The equations of motion for point vortices on the sphere are derived in [6, 17, 28]. Like the planar case, these equations are Hamiltonian and this property has been used to study them from a number of different viewpoints. Phase space reduction shows that the three vortex problem is completely integrable on both the plane and the sphere (for references see [14]). A detailed analysis of its dynamics on the sphere is given in [14, 15] and streamlines for the corresponding fluid flows on the sphere are studied in [16]. The energy-momentum method is applied to the stability of the relative equilibria of the three vortex problem in [37]. Much less is known about systems of  $N$  point vortices when  $N$  is greater than three. Results on the topology of reduced phase spaces and estimates of the numbers of relative equilibria are given in [19]. Phase space reduction techniques have also been described in [21] and [37]. The stability of a ring of  $N$  identical point vortices is studied in [38] and the results extended to systems of finite area vortices, while the stability of relative equilibria for 3 point vortices is described in [37]. Finally [22, 23] discuss the statistical mechanics of systems of point vortices on the sphere. OK? Anything else?

Our main aim in this paper is to show that the symmetry properties of the point vortex equations of motion on the sphere can be exploited to obtain detailed information about relative equilibria for any value of  $N$ . The results include a classification of possible types of relative equilibria, a number of existence theorems and the construction of bifurcation diagrams for some specific cases. A sequel to this paper [24] will give an analogous discussion of the stability properties of some of the relative equilibria described in this paper. Applications to geophysical fluid dynamics will be discussed in [25]. OK?!

Point vortex systems on the sphere have three different types of symmetries. The Hamiltonian  $H$  is invariant under rotations of the sphere, reflections of the sphere and permutations of identical vortices. The rotations of the sphere define an action of the three dimensional rotation group  $\mathbf{SO}(3)$  on the phase space  $\mathcal{P}$ . The reflections extend this to an action of the full orthogonal group  $\mathbf{O}(3)$  and the permutations define an action of a finite group  $S$ . Putting all these together gives an action of the group  $\mathbf{O}(3) \times S$ . We denote this group by  $\widehat{G}$  and the subgroup  $\mathbf{SO}(3) \times S$  by  $G$ .

The group  $G = \mathbf{SO}(3) \times S$  acts *symplectically* on the phase space and this fact, combined with the invariance of the Hamiltonian, implies that the equations of motion are equivariant. This means that they restrict to any submanifold of  $\mathcal{P}$  which is the fixed point set of a subgroup  $\Sigma < \mathbf{SO}(3) \times S$ . Such fixed point spaces are symplectic submanifolds of  $\mathcal{P}$  and the restricted equations of motion are again Hamiltonian, with Hamiltonian equal to the restriction of the full Hamiltonian  $H$  to the fixed point set. This ‘discrete reduction’ technique (see eg [27]) is especially useful for analysing high dimensional systems with large symmetry groups. In particular critical points of the restricted Hamiltonian are equilibrium points of the restricted flow and hence also equilibrium points for the full flow.

It is important to notice that discrete reduction does not work for general subgroups  $\Sigma$  of the full symmetry group  $\widehat{G} = \mathbf{O}(3) \times S$ . The reflections in  $\mathbf{O}(3)$  act *anti-symplectically* on  $\mathcal{P}$  (see §1.2) and are therefore *time-reversing* symmetries of the equations of motion. Simple examples show that fixed point sets of such time-reversing symmetries are not in general invariant submanifolds for the equations of motion. However

critical points of the restriction of the Hamiltonian  $H$  to such a fixed point set *are* critical points of the full Hamiltonian and so are also equilibrium points of the full flow. Thus ‘discrete reduction by time-reversing symmetries’ can be used to find equilibria. This is an example of the *principle of symmetric criticality* [34] in action.

This simple observation extends to *relative equilibria*. These are orbits of the group action which are invariant under the flow and correspond to motions of the point vortices which are stationary in some steadily rotating frame. Noether’s theorem to the action of  $\mathbf{SO}(3)$  on  $\mathcal{P}$  shows that the flow preserves the level sets of the ‘centre of vorticity’ map (or, momentum map)  $\Phi : \mathcal{P} \rightarrow \mathbf{R}^3$ , and relative equilibria are critical points of the restrictions  $H_\mu$  of  $H$  to the level sets  $\Phi^{-1}(\mu)$ . The functions  $H_\mu$  are invariant under the subgroups  $\widehat{G}_\mu$  of  $\mathbf{O}(3) \times S$  which preserve  $\Phi^{-1}(\mu)$  and the critical points of their restrictions to fixed point sets of subgroups of  $\widehat{G}_\mu$  are also critical points of  $H_\mu$  and hence relative equilibria.

We refer to the largest subgroup of  $\widehat{G}$  that fixes a particular configuration of vortices as its *symmetry group*  $\Sigma$ . Group theoretically this is the isotropy subgroup of the corresponding point  $\mathbf{x} \in \mathcal{P}$ :

$$\Sigma = \widehat{G}_{\mathbf{x}} = \{g \in \widehat{G} : g \cdot \mathbf{x} = \mathbf{x}\}.$$

Points which lie in the same orbits of the action of  $\widehat{G}$  on  $\mathcal{P}$  have conjugate isotropy subgroups and the set of points in  $\mathcal{P}$  which have isotropy subgroups conjugate to a given subgroup of  $\widehat{G}$  is called an *orbit type set*, and each connected component of this orbit type set is called *orbit type stratum*. Inclusion of one subgroup in another induces a partial ordering on the set of conjugacy classes of isotropy subgroups of a group action which is closely related to the partial ordering on the set of orbit type strata given by inclusion of one stratum in the closure of another. These partial orderings are frequently referred to as *orbit type lattices* and they play an important role in the bifurcation theory of systems with symmetries. See for example [10] and references therein. The strata which are minimal in these orderings, ie those which are closed in  $\mathcal{P}$ , play a particularly important role [1, 29, 30]. A compact stratum must contain at least one critical point of any  $\widehat{G}$ -invariant function on  $\mathcal{P}$ , and the same will be true for non-compact minimal strata if the function is unbounded as it approaches the ‘boundaries’ of the stratum.

Much of this paper is dedicated to fully exploiting these observations (though hopefully not *ad nauseam*). In Section 1.3 we give a detailed description of the symmetry groups  $G$  and  $\widehat{G}$  and classify configurations of points vortices on the sphere according to their isotropy subgroups under the actions of these groups on the phase space  $\mathcal{P}$ . The symmetry types of configurations which lie in any given level set  $\Phi^{-1}(\mu)$  are also discussed. The *adjacency* of one orbit type stratum to another is defined and a method of computing adjacencies described. As an illustration of the general method the results are applied to the special case of  $N$  identical vortices. In particular Table 3 and Proposition 1.12 give the minimal strata for, respectively, the actions of  $G$  and  $\widehat{G}$  on  $\mathcal{P}$  for arbitrary  $N$ . Propositions 1.14 and 1.15 together describe the symmetry types of configurations which lie in each level set  $\Phi^{-1}(\mu)$ , again for arbitrary  $N$ . Table 4 lists the minimal strata for the actions of  $G$  and  $\widehat{G}$  on  $\mathcal{P}$ , and also the  $\widehat{G}$  strata which are minimal in  $\Phi^{-1}(\mu)$  when  $\mu \neq 0$ , for  $N = 3 \dots 12$ . Finally Figures 1 and 2 give the complete orbit type lattices for the  $G$  and  $\widehat{G}$  actions on  $\mathcal{P}$  for  $N$  up to 6 and 5, respectively.

The results on the orbit type stratifications of the actions of  $G$  and  $\widehat{G}$  on  $\mathcal{P}$  are purely group theoretical and are applicable to other problems which can be formulated in terms of finite sets of ‘weighted’ points on the sphere. Obvious examples include equilibrium configurations of sets of charged particles on the sphere (see eg [9]) and perhaps also models of superconductors on the sphere [8]. Closely related problems include the calculation of isotropy subgroups of representations of the groups  $\mathbf{SU}(2)$  and  $\mathbf{SO}(3)$  [5, 13]. Points in these representation spaces can be represented by homogenous complex polynomials in two variables. The roots of such a polynomial of degree  $N$  associate to it a set of  $N$  points in complex projective space, and hence on the sphere. A important difference in this case is that repeated roots are not excluded and so ‘collisions’ of the particles are allowed.

In Section 2 we state the main theorems of the paper. Theorem 2.2 gives sufficient conditions for the existence of equilibrium points in minimal strata while Theorem 2.5 gives the analogous result for the existence of relative equilibria in minimal strata in  $\Phi^{-1}(\mu)$ . Section 2.3 contains two results on the existence of families of relative equilibria, parameterized by their centres of vorticity  $\mu$ , which bifurcate from (relative) equilibria with  $\mu = 0$ . The methods used are adapted from an analogous treatment of relative equilibria of molecules [32]. Table 5 summarizes the main results for bifurcations from equilibrium.

The results of Section 2 are applicable to any Hamiltonian system on  $\mathcal{P}$  which has the same symplectic form and symmetries as the point-vortex Hamiltonian. However in §3 we restrict attention to the point-vortex Hamiltonian itself. The results of §2.2 imply that any relative equilibrium that is not an equilibrium must consist of a number,  $k$  say, of ‘latitudinal’ rings of vortices with the same number,  $n$ , of vortices in each and such that all the vortices in the same rings have the same vorticity. In addition there may be  $\ell = 1$  or 2 ‘polar’ vortices. Such configurations are denoted  $C_n(kR, \ell p)$ . These configurations are analogous to the vortex polygons that have been extensively studied in the plane (see [20] and references therein). In Proposition 3.3 we give equations for such a configuration to be a relative equilibrium and then discuss the solutions of these equations when  $k = 1$  and  $k = 2$ . In particular we show (Proposition 3.6) that when  $k = 2$  the two rings must be either ‘aligned’ or ‘staggered’ with respect to each other; these are denoted  $C_{nv}(2R)$  and  $C_{nv}(R, R')$  respectively. Some numerical observations of how these relative equilibria can bifurcate as the centre of vorticity is varied are also briefly reported on. Section 3.2 contains an existence and uniqueness theorem for equilibrium solutions with the vortices arranged round the equator. It is hoped that convexity argument used to prove this might be extended to more general relative equilibria.

Finally, in Section 4, we look at the relative equilibria of  $N$  identical vortices when  $N = 3, 4, 5$  and 6 and in particular show how the methods developed in the paper can be used to begin the construction of energy-momentum relative equilibrium bifurcation diagrams, Figures 3, 4, 5 and 6. These include all the relative equilibria of all the symmetry types that are predicted by the results of Section 2. Comparison with the orbit type lattices in figures 1 and 2 show that relative equilibria of many of the orbit types do occur. The discussion in Section 4 and these figures also summarize the non-existence results we have been able to deduce here or, in the case of  $N = 3$ , taken from [14]. However, we know that for  $N > 3$  our bifurcation diagrams are incomplete. Several arguments, summarized in Section 4, indicate that there must be further lower symmetry relative equilibria that we have not yet found. Work on the stability of relative equilibria and these bifurcation diagrams will continue and be reported on in [24].

## Acknowledgements

This work was supported by EPSRC Visiting Fellowship grants to support visits by Lim and Montaldi to the University of Warwick and by an EPSRC Research Grant. Anything else?

## 1 Symmetries of Vortex Systems on $S^2$

In this section we describe the symmetries of the equations of motion of systems of point vortices on the sphere  $S^2$  and classify the possible configurations of vortices by their isotropy subgroups. Particular attention is paid to time-reversing symmetries.

## 1.1 Equations of motion

Let  $\overline{\mathcal{P}} = S^2 \times \dots \times S^2$ , the product of  $N$  copies of the unit sphere in  $\mathbf{R}^3$ . The phase space for the  $N$ -vortex problem is  $\mathcal{P} = \overline{\mathcal{P}} \setminus \Delta$ , where  $\Delta$  is the ‘big diagonal’ where at least two points coincide:

$$\Delta = \{\mathbf{x} = (x_1, \dots, x_N) \mid x_i = x_j \text{ for some } i \neq j\}.$$

The symplectic structure on  $\mathcal{P}$  is given by

$$\omega = \sum \lambda_j \pi_j^* \omega_{S^2}, \quad (1.1)$$

where  $\pi_j$  is the Cartesian projection on to the  $j$ th factor,  $\omega_{S^2}$  is the natural symplectic form on  $S^2$ , and  $\lambda_j$  is the vorticity of the  $j$ -th vortex. The Poisson structure is given by

$$\{f, g\} = \sum_j \lambda_j^{-1} [d_j f, d_j g, x_j],$$

where  $[d_j f, d_j g, x_j]$  is the triple product  $[a, b, c] = a \cdot (b \times c)$  for  $a, b, c \in \mathbf{R}^3$ . The Hamiltonian function is:

$$H(x_1, \dots, x_N) = - \sum_{i < j} \lambda_i \lambda_j \log(1 - x_i \cdot x_j) \quad (1.2)$$

and the Hamiltonian vector field  $X_H$  is given by

$$\dot{x}_j = X_H(\mathbf{x})_j = \sum_{i \neq j} \lambda_i \frac{x_i \times x_j}{1 - x_i \cdot x_j}. \quad (1.3)$$

As usual, the relationship between the Poisson structure, Hamiltonian and Hamiltonian vector field is given by:

$$\frac{df}{dt} = df \cdot X_H = \{f, H\}$$

for any function  $f$ .

## 1.2 Symmetries of the equations

Let  $\Lambda = (\lambda_1, \dots, \lambda_N)$  be the ordered collection of vorticities of the  $N$  vortices. Denote by  $S(\Lambda)$  the subgroup of the symmetric group  $S_N$  which preserves  $\Lambda$ :

$$S(\Lambda) = \{\sigma \in S_N \mid \lambda_{\sigma(j)} = \lambda_j \text{ for all } j\}.$$

We will often assume that all the vorticities are equal, so that  $S(\Lambda) = S_N$ .

Let  $G = \mathbf{SO}(3) \times S(\Lambda)$ , and  $\widehat{G} = \mathbf{O}(3) \times S(\Lambda)$ . Then  $G$  and  $\widehat{G}$  act on the phase space  $\mathcal{P}$  by

$$(A, \sigma)(x_1, \dots, x_N) = (Ax_{\sigma(1)}, \dots, Ax_{\sigma(N)}). \quad (1.4)$$

If we define  $\det(A, \sigma) = \det(A)$  then  $G = \det^{-1}(1)$ . Note that  $G$  acts by symplectic transformations, while elements of  $\widehat{G} \setminus G$  act *anti-symplectically*, that is for  $g \in \widehat{G}$ ,

$$\omega_{g \cdot \mathbf{x}}(dg_{\mathbf{x}}(v), dg_{\mathbf{x}}(w)) = \det(g) \omega_{\mathbf{x}}(v, w) \quad (1.5)$$

for all  $v, w$  in  $T_{\mathbf{x}}\mathcal{P}$ .

**Proposition 1.1**

1. The Hamiltonian (1.2) for the  $N$ -vortex problem with vortex set  $\Lambda$  is invariant under the action of  $\widehat{G}(\Lambda)$  on  $\mathcal{P}$ .
2. The vector field  $X_H$  given by (1.3) is  $\widehat{G}$  semi-equivariant: for  $g \in \widehat{G}$  it satisfies

$$X_H(g \cdot \mathbf{x}) = \det(g) dg_{\mathbf{x}}(X_H(\mathbf{x})). \quad (1.6)$$

PROOF Statement (i) is clear while the second follows from (i), together with the fact that  $g \in G$  acts symplectically on  $\mathcal{P}$  while  $g \in \widehat{G} \setminus G$  acts anti-symplectically.  $\square$

The transformation property (1.6) implies that the elements  $g \in \widehat{G} \setminus G$  are *time reversing* symmetries of  $X_H$ : if  $t \mapsto \mathbf{x}(t)$  is the trajectory with initial value  $\mathbf{x}_0$ , then the trajectory with initial value  $g(\mathbf{x}_0)$  is  $t \mapsto g(\mathbf{x}(-t))$ . This situation, resulting from a group acting by a combination of symplectic and anti-symplectic symmetries, can be formalised as follows. Let  $\widehat{G}$  be any group and  $\chi : \widehat{G} \rightarrow \mathbf{Z}_2 = \{1, -1\}$  a group homomorphism with kernel  $G$ . Then an action of  $\widehat{G}$  on a symplectic manifold  $(\mathcal{P}, \omega)$  is said to be *semisymplectic* with temporal character  $\chi$ , if

$$\omega_{g \cdot \mathbf{x}}(dg(v), dg(w)) = \chi(g) \omega_{\mathbf{x}}(v, w). \quad (1.7)$$

In the case of the  $N$ -vortex problem the homomorphism  $\chi$  is simply given by  $\chi(A, \sigma) = \det(A)$ . For more details see [33]

Hamiltonian systems with continuous symmetry groups satisfy conservation laws (Noether's theorem). These conserved quantities are the components of the *momentum map*  $\Phi$ .

**Proposition 1.2**

1. After identifying the Lie algebra dual  $\mathfrak{so}(3)^*$  with  $\mathbf{R}^3$  in 'the usual way', the momentum map for the  $\mathbf{SO}(3)$  action is

$$\Phi(x_1, \dots, x_N) = \sum_{j=1}^N \lambda_j x_j.$$

2. Let  $\pi : \widehat{G} \rightarrow \mathbf{O}(3)$  be the Cartesian projection. Then  $\Phi$  is  $\widehat{G}$  equivariant, with  $\widehat{G}$  acting on  $\mathfrak{so}(3)^* = \mathbf{R}^3$  via the representation  $\pi$ .

The 'usual' identification of  $\mathfrak{so}(3)^*$  with  $\mathbf{R}^3$  relates the skew-symmetric matrix  $\mu$  with the vector  $\hat{\mu}$  which satisfies  $\mu a = \hat{\mu} \times a$ , for all  $a \in \mathbf{R}^3$ . This is identical to the 'usual' identification of  $\mathfrak{so}(3)$  with  $\mathbf{R}^3$ ,  $\xi \mapsto \hat{\xi}$ , with  $\xi a = \hat{\xi} \times a$ . The standard pairing of  $\mathfrak{so}(3)^*$  with  $\mathfrak{so}(3)$  given by  $\langle \mu, \xi \rangle = \frac{1}{2} \text{tr}(\mu^T \xi)$  becomes the standard pairing of  $\mathbf{R}^3$  with itself:  $\langle \hat{\mu}, \hat{\xi} \rangle = \mu \cdot \xi$ .

PROOF For the first part, from the definition of momentum maps we need to show that  $\{f, \Phi_{\xi}\}(\mathbf{x}) = df_{\mathbf{x}} \cdot \xi_{\mathcal{P}}(\mathbf{x})$  for each element  $\xi \in \mathfrak{so}(3)$ . The expression for  $\Phi$  in the proposition gives

$$\Phi_{\xi}(\mathbf{x}) = \sum_j \lambda_j \hat{\xi} \cdot \mathbf{x}_j$$

and so, by definition of the Poisson structure,

$$\{f, \Phi_\xi\}(\mathbf{x}) = \sum_j \lambda_j^{-1} (d_j \Phi_\xi \times \mathbf{x}_j) \cdot d_j f_{\mathbf{x}} \quad (1.8)$$

$$= \sum_j \lambda_j^{-1} (\lambda_j \hat{\xi} \times \mathbf{x}_j) \cdot d_j f_{\mathbf{x}} \quad (1.9)$$

$$= d_j f_{\mathbf{x}} \cdot \xi_{\mathcal{P}}(\mathbf{x}), \quad (1.10)$$

as required. The second statement in the proposition follows immediately from the formula for  $\Phi$ .  $\square$

We call  $\Phi(\mathbf{x})$  the *centre of vorticity* of the configuration  $\mathbf{x}$ .

**Remark 1.3** For certain collections of vorticities, there are also anti-symplectic permutations, namely those permutations  $\sigma \in S_N$  for which  $\lambda_{\sigma(j)} = -\lambda_j$  for all  $j$ . Such permutations preserve the Hamiltonian function and therefore act on the vector field by time-reversing symmetries. Let  $\hat{S}(\Lambda)$  be the group of all permutations for which there is a  $\chi(\sigma) = \pm 1$  such that  $\lambda_{\sigma(j)} = \chi(\sigma)\lambda_j$  for all  $j$ . Then the full symmetry group of the Hamiltonian is  $\mathbf{O}(3) \times \hat{S}(\Lambda)$ . An element  $(A, \sigma) \in \mathbf{O}(3) \times \hat{S}(\Lambda)$  is symplectic or anti-symplectic if the sign of  $\chi(\sigma) \det(A)$  is, respectively, positive or negative. In this paper we are principally interested in the case where all the vortices are identical, so we do not pursue this any further.  $\diamond$

### 1.3 Isotropy subgroups and the stratification

Recall that if  $G$  is a Lie group acting on a manifold  $\mathcal{P}$ , then the *isotropy subgroup* of a point  $\mathbf{x}$  is

$$G_{\mathbf{x}} = \{g \in G \mid g \cdot \mathbf{x} = \mathbf{x}\}.$$

The conjugacy class of  $G_{\mathbf{x}}$  is called the *orbit type* of  $\mathbf{x}$ , and the set of points of given orbit type is a union of smooth submanifolds, whose connected components are called *orbit type strata*. The set of all strata is a partition of  $\mathcal{P}$  known as an *orbit type stratification*. Furthermore, if  $\Sigma < G$  then the  $\Sigma$  fixed-point set is

$$\text{Fix}(\Sigma, \mathcal{P}) = \{\mathbf{x} \in \mathcal{P} \mid \sigma \cdot \mathbf{x} = \mathbf{x}, \forall \sigma \in \Sigma\}.$$

This is a closed submanifold of  $\mathcal{P}$  containing the strata of orbit type  $\Sigma$ .

We will be working with a certain refinement of the notion of orbit type, which we call *point-orbit type*. For most strata the two coincide; see Remark 1.4.

In the remainder of this section we compute the possible  $G$  and  $\hat{G}$  orbit types of configurations of point vortices on the sphere for any given  $N$  and  $\Lambda$ . The corresponding orbit type strata are used in Section 2 to prove the existence of symmetric (relative) equilibria, while the adjacencies between the strata are used to study bifurcations.

Note that  $G_{\mathbf{x}} = \hat{G}_{\mathbf{x}} \cap G$  and that  $\hat{G}_{\mathbf{x}}$  is either equal to  $G_{\mathbf{x}}$  or is an extension of order two. We also say that  $\hat{G}_{\mathbf{x}}$  and  $G_{\mathbf{x}}$  are the *symmetry groups* of  $\mathbf{x}$ . We are sometimes careless about distinguishing between a subgroup and its conjugacy class, and write  $\Gamma = \Gamma'$  when we mean that  $\Gamma$  and  $\Gamma'$  are conjugate.

Each  $G$  orbit type stratum is a union of  $\hat{G}$  orbit type strata, so the  $\hat{G}$  orbit type stratification refines the  $G$  orbit type stratification. This is readily seen by comparing Figures 1 and 2 (pages 16 and 17).

There is a very important difference between the  $G$  and  $\hat{G}$  orbit type strata. Because  $G$  acts symplectically the set of points with isotropy subgroup  $G_{\mathbf{x}}$  (for some  $\mathbf{x}$ ) is a symplectic submanifold that is invariant under the flow generated by any  $G$ -invariant Hamiltonian. And since the  $G$  orbit type strata are unions of

such submanifolds, they too are invariant under such flows. This is not true for the  $\widehat{G}$  orbit type strata: trajectories with initial conditions in one stratum do not necessarily stay in that stratum. However in the next section we show that  $\widehat{G}$  invariant Hamiltonians will necessarily have some trajectories, and in particular relative equilibria, which do stay in the same  $\widehat{G}$  stratum for all time, though not within a given fixed point space.

In order to list all the possible orbit type strata, we make use of the following observation. Suppose  $\Sigma < \widehat{G}$  fixes a point  $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{P}$ . Since the points  $x_1, \dots, x_N$  are distinct, it follows that the permutation group  $S(\Lambda)$  acts freely on  $\mathcal{P}$ , and so  $\Sigma$  is isomorphic to its projection  $\Gamma$  into  $\mathbf{O}(3)$ , that is  $\Sigma \cong \Gamma = \pi(\Sigma) < \mathbf{O}(3)$ . Equivalently, any such subgroup  $\Sigma < \widehat{G}$  can be reconstructed from its projection  $\Gamma = \pi(\Sigma)$  as the graph of a homomorphism  $\Gamma \rightarrow S(\Lambda)$ . Furthermore, we shall always assume that  $N > 2$ , which implies that the symmetry group of any configuration is finite. The subgroup  $\Gamma < \mathbf{O}(3)$  acts on the set of point vortices by permuting them, and this set decomposes into finitely many irreducible sets, or orbits, which we call *point-orbits* (to distinguish from the orbits of  $\widehat{G}$  in  $\mathcal{P}$ ). An analogous observation holds for subgroups of  $G$ , with  $\mathbf{SO}(3)$  replacing  $\mathbf{O}(3)$ .

Rather than looking directly for these isotropy subgroups we start by describing all the possible configurations which are fixed by each finite subgroup of  $\mathbf{O}(3)$ , and hence the values of  $N$  and  $\Lambda$  for which each of these fixed point sets is non-empty. For any given  $N$  and  $\Lambda$  it is then relatively straightforward to determine all the possible isotropy subgroups. In each subsection below, we first consider the more straightforward case of finite subgroups of  $\mathbf{SO}(3)$  before continuing with the more general case of finite subgroups of  $\mathbf{O}(3)$  and the  $\widehat{G}$ -orbit types.

**Tables 1 and 2** The finite subgroups of  $\mathbf{SO}(3)$  and  $\mathbf{O}(3)$  are listed in Tables 1 and 2 respectively, along with a classification of their possible point-orbits  $\mathcal{O}$  in  $S^2$ . The first column in each table lists the groups, using the usual Schönflies notation. The second column gives the labels we use to identify the different types of point-orbit of the action of that group on  $S^2$ . The third column gives the isotropy subgroup for the action at a point in that point-orbit and the fourth column the number of points in the point-orbit. The fifth column gives the dimension of the set of each type of point-orbit, and in the case that the dimension is zero, in parentheses the maximum number of point-orbits of that type that the group has in  $S^2$ . The final column gives a brief description of the point-orbit. An  $n$ -ring is a regular  $n$ -gon. ‘Vertical’ refers to the axis of the rotation subgroup in each case. ‘Vertically aligned’ means that the vortices of one ring are directly above those of the other ring, and ‘vertically staggered’ means that the upper one is rotated by  $\pi/n$  with respect to the lower. For more details see the appendix, and for more on  $R, R', r, r'$  etc., see Remark 1.4. Readers unfamiliar with the Schönflies notation should consult the appendix, which contains a description of each subgroup together with their different point-orbits.

For each subgroup of  $\mathbf{O}(3)$  the finite sets of points in  $S^2$  which are invariant under the group are finite unions of the point-orbits listed in Tables 1 and 2. For example, if there are 54 identical point vortices then there is a configuration with octahedral symmetry; namely 6 vortices at the vertices of the octahedron (point-orbit type  $v$ ), together with two sets of 24 vortices generated by any general point on the sphere (point-orbit type  $R$ ).

**Notation** It follows that for any given  $N$  and  $\Lambda$  the isotropy subgroups of  $\widehat{G}$  on  $\mathcal{P}$  correspond exactly to the isotropy subgroups of the action of  $\mathbf{O}(3)$  (or  $\mathbf{SO}(3)$ ) on finite sets of points in  $S^2$  labelled by their vorticities. Exactly the same statement holds for the isotropy subgroups of  $G$ , except that  $\mathbf{O}(3)$  is replaced by  $\mathbf{SO}(3)$ . We may therefore denote the orbit types by symbols of the form

$$\Gamma(k_1\mathcal{O}_1, k_2\mathcal{O}_2, \dots, k_r\mathcal{O}_r),$$

$\Gamma$	$\mathcal{O}$	$K$	$ \mathcal{O} $	Dim	Description
$\mathbf{C}_n$	$R$	1	$n$	2	$n$ -ring
	$p$	$\mathbf{C}_n$	1	0(2)	pole
$\mathbf{D}_n$	$R$	1	$2n$	2	pair of $n$ -rings on opposite latitudes
	$r$	$\mathbf{C}_2$	$n$	0(2)	equatorial $n$ -ring or dual
	$p$	$\mathbf{C}_n$	2	0(1)	pair of poles
$\mathbb{T}$	$R$	1	12	2	regular $\mathbb{T}$ orbit
	$e$	$\mathbf{C}_2$	6	0(1)	mid-points of edges of tetrahedron
	$v$	$\mathbf{C}_3$	4	0(2)	vertices of tetrahedron or dual
$\mathbb{O}$	$R$	1	24	2	regular $\mathbb{O}$ orbit
	$e$	$\mathbf{C}_2$	12	0(1)	mid-points of edges of octahedron
	$f$	$\mathbf{C}_3$	8	0(1)	mid-points of faces of octahedron
	$v$	$\mathbf{C}_4$	6	0(1)	vertices of octahedron
$\mathbb{I}$	$R$	1	60	2	regular $\mathbb{I}$ orbit
	$e$	$\mathbf{C}_2$	30	0(1)	mid-points of edges of icosahedron
	$f$	$\mathbf{C}_3$	20	0(1)	mid-points of faces of icosahedron
	$v$	$\mathbf{C}_5$	12	0(1)	vertices of icosahedron

 Table 1: Classification of point-orbits of finite subgroups of  $\mathbf{SO}(3)$ . See text for explanations

which we call *point-orbit type symbols* where  $\Gamma$  is the projection of the isotropy subgroup into  $\mathbf{O}(3)$  (or  $\mathbf{SO}(3)$ ), and the terms in parentheses denote the way in which  $\Gamma$  acts on the finite set of point vortices: there are  $k_1$  orbits (or point-orbits) of type  $\mathcal{O}_1$ ,  $k_2$  of type  $\mathcal{O}_2$  etc. We also tacitly assume that for  $i \neq j$  the point-orbit types  $\mathcal{O}_i$  and  $\mathcal{O}_j$  are distinct. For example, if the isotropy subgroup in question is the orientation preserving symmetry group  $\mathbb{O}$  of the cube, and if the vortices lie in the configurations described above, then the symmetry type is denoted  $\mathbb{O}(2R, v)$ .

Notice that the notation  $\Gamma(k_1\mathcal{O}_1, k_2\mathcal{O}_2, \dots, k_r\mathcal{O}_r)$  does indeed determine the *orbit type* of a particular configuration, since  $\Gamma$  is a finite subgroup of  $\mathbf{SO}(3)$  or  $\mathbf{O}(3)$  *up to conjugacy*, and the orbits  $\mathcal{O}_1, \dots, \mathcal{O}_r$  describe how the elements of  $\Gamma$  act by permuting the set of vortices, but without labelling the vortices — so specifying the homomorphism  $\Gamma \rightarrow S_N$ , again *up to conjugacy*.

Of course  $\Gamma(k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d)$  will only occur as an orbit type for certain  $N$  and  $\Lambda$ . Specifically  $N$  must be equal to  $k_1|\mathcal{O}_1| + \dots + k_d|\mathcal{O}_d|$  and  $\Lambda$  must assign the same vorticities to all the points in the same point-orbit of  $\Gamma$ . Furthermore, certain point-orbit type symbols do not correspond to isotropy subgroups; see Proposition 1.6 below.

**Remark 1.4** There is one important point that merits further explanation. For the  $G$ -action, the orbit type and the point-orbit type are equivalent, but for the  $\widehat{G}$ -action the point-orbit type refines the  $\widehat{G}$  orbit type in a few cases.

The basic case is that of the symmetry type  $\mathbf{C}_{nv}$ , whose point-orbits are each of the poles, certain ‘horizontal’ regular  $n$ -gons ( $n$ -rings) and horizontal semi-regular  $2n$ -gons. We denote a given ring by  $(R)$  and its dual by  $(R')$ . If  $n$  is even, then  $(R)$  and  $(R')$  in fact have non-conjugate (although isomorphic) symmetry: for example, a square and its dual have the same symmetry group, but it permutes the vertices in different ways. Thus  $\mathbf{C}_{nv}(2R)$  and  $\mathbf{C}_{nv}(R, R')$  need to be distinguished on grounds of pure symmetry. However, if  $n$  is odd, a ring and its dual have *conjugate* permutational symmetry, but we still need to distinguish between  $\mathbf{C}_{nv}(2R)$  and  $\mathbf{C}_{nv}(R, R')$  as they are geometrically distinct — the first consists of a

$\Gamma$	$\mathcal{O}$	$K$	$ \mathcal{O} $	Dim	Description
$C_{nv}$	$R_s$	1	$2n$	2	semi-regular $2n$ -gon
	$R, R'$	$C_h$	$n$	1	regular $n$ -ring or dual
	$p$	$C_{nv}$	1	0(2)	pole
$C_{nh}$	$R$	1	$2n$	2	pair of $n$ -rings on opposite latitudes
	$R^e$	$C_h$	$n$	1	equatorial $n$ -ring
	$p$	$C_n$	2	0(1)	pair of poles
$D_{nh}$	$R_s$	1	$4n$	2	vertically aligned pair of semi-regular $2n$ -gons
	$R_s^e$	$C_h$	$2n$	1	equatorial semi-regular $2n$ -gon
	$R, R'$	$C'_h$	$2n$	1	vertically aligned pair of $n$ -rings or duals
	$r, r'$	$C_{2v}$	$n$	0(1)	equatorial $n$ -ring or dual
	$p$	$C_{nv}$	2	0(1)	pair of poles
$D_{nd}$	$R_s$	1	$4n$	2	vertically staggered pair of semi-regular $2n$ -gons
	$R$	$C_h$	$2n$	1	vertically staggered pair of $n$ -rings
	$r$	$C_2$	$2n$	0(1)	equatorial $2n$ -ring
	$p$	$C_{nv}$	2	0(1)	pair of poles
$S_{2n}$	$R$	1	$2n$	2	vertically staggered pair of $n$ -rings
	$p$	$C_n$	2	0(1)	pair of poles
$C_h$	$R$	1	2	2	vertically aligned pair of points
	$E$	$C_h$	1	1	equatorial point
$C_i$	$R$	1	2	2	pair of antipodal points
$T_d$	$R$	1	24	2	regular $T_d$ orbit
	$E$	$C_h$	12	1	generic orbit on edges of tetrahedron
	$e$	$C_{2v}$	6	0(1)	mid-points of edges of tetrahedron
	$v$	$C_{3v}$	4	0(2)	vertices of tetrahedron or dual
$T_h$	$R$	1	24	2	regular $T_h$ orbit
	$E$	$C_h$	12	1	generic orbit on 'equator'
	$e$	$C_{2v}$	6	0(1)	mid-points of edges of tetrahedron
	$v$	$C_3$	8	0(1)	vertices of cube
$O_h$	$R$	1	48	2	regular $O_h$ orbit
	$E$	$C_h$	24	1	generic orbit on edges of octahedron
	$E'$	$C'_h$	24	1	generic orbit on face bisectors of octahedron
	$e$	$C_{2v}$	12	0(1)	mid-points of edges of octahedron
	$f$	$C_{3v}$	8	0(1)	mid-points of faces of octahedron
	$v$	$C_{4v}$	6	0(1)	vertices of octahedron
$I_h$	$R$	1	120	2	regular $I_h$ orbit
	$E$	$C_h$	60	1	generic orbit on edges of icosahedron
	$e$	$C_{2v}$	30	0(1)	mid-points of edges of icosahedron
	$f$	$C_{3v}$	20	0(1)	mid-points of faces of icosahedron
	$v$	$C_{5v}$	12	0(1)	vertices of icosahedron

Table 2: Classification of point-orbits of finite subgroups of  $O(3)$ . See text for explanations

pair of  $n$ -rings which are vertically aligned (on the same longitudes) while in the second they are ‘vertically staggered’ (on intermediate longitudes), moreover  $\mathbf{C}_{nv}(2R)$  specializes to  $\mathbf{D}_{nh}(R)$  when the two rings are on opposite latitudes, while  $\mathbf{C}_{nv}(R, R')$  specializes under the same circumstances to  $\mathbf{D}_{nd}(R)$ . In fact the two are different connected components (strata) of the same symmetry type sets in the orbit space  $\mathcal{P}/\widehat{G}$ , as follows from Theorem 1.10. Since strata are by definition connected, it follows that the resulting stratifications by orbit type and by point-orbit type coincide.

Similar remarks are valid for  $\mathbf{D}_{nh}(2R)$  and  $\mathbf{D}_{nh}(R, R')$ , as well as for more complicated combinations of rings, such as  $\mathbf{C}_{nv}(3R, 2R', p)$  compared with  $\mathbf{C}_{nv}(5R, p)$ , or  $\mathbf{D}_{nh}(R, r, p)$  compared with  $\mathbf{D}_{nh}(R, r', p)$ .

On the other hand, for the tetrahedral groups  $\mathbb{T}$  and  $\mathbb{T}_d$  there is the point-orbit consisting of the vertices of the tetrahedron, which we denote  $(v)$ , and there is the point-orbit consisting of the vertices of the dual tetrahedron, say  $(v')$ . If there are 8 vortices, occupying both sets of vertices, we write  $\mathbb{T}(2v)$  or  $\mathbb{T}_d(2v)$ , rather than  $\mathbb{T}(v, v')$  or  $\mathbb{T}_d(v, v')$  because there is no ambiguity: in  $\mathbb{T}(2v)$  the two point-orbits cannot coincide, so they must be dual. The same remark is valid for the poles under cyclic  $\mathbf{C}_n$  or  $\mathbf{C}_{nv}$  symmetry: we write  $\mathbf{C}_n(2p)$  rather than  $\mathbf{C}_n(p, p')$ , and similarly for  $\mathbf{C}_{nv}$ .  $\diamond$

The following proposition can be established by a lengthy inspection of Table 1.

**Proposition 1.5** *If all the vorticities are identical and  $N > 2$  then for any finite subgroup  $\Gamma < \mathbf{SO}(3)$  there exists an isotropy subgroup  $\Sigma$  with  $\Gamma < \pi(\Sigma)$  if and only if*

$$N = \begin{cases} kn + \{0, 1, 2\} & \text{if } \Gamma = \mathbf{C}_n \\ 2kn + \{0, 2, n, n+2\} & \text{if } \Gamma = \mathbf{D}_n \\ 12k + \{0, 4, 6, 8, 10, 12, 14\} & \text{if } \Gamma = \mathbb{T} \\ 24k + \{0, 6, 8, 12, 14, 18, 20, 26\} & \text{if } \Gamma = \mathbb{O} \\ 60k + \{0, 12, 20, 30, 32, 42, 50, 62\} & \text{if } \Gamma = \mathbb{I} \end{cases} \quad (1.11)$$

where  $k$  is always a non-negative integer.

A similar result can be obtained for the  $\widehat{G}$  action; the details are left to the reader.

It should be pointed out that not all point-orbit type symbols are the symbols of isotropy strata, whence the inclusion  $\Gamma < \pi(\Sigma)$  in the proposition above. For example, any configuration with symmetry  $\mathbf{S}_{2n}(R)$  in fact has symmetry  $\mathbf{D}_{2d}(R)$ , although for a given representative of the conjugacy class  $\mathbf{S}_{2n}(R)$  there are infinitely many supergroups of type  $\mathbf{D}_{2d}(R)$ , and which one is the symmetry group depends on the configuration in question. On the other hand,  $\mathbf{S}_{2n}(2R)$  is an isotropy type.

**Proposition 1.6** *For  $N > 2$  identical vortices, all point-orbit symbols correspond to isotropy strata, with the following exceptions:*

1. *For the  $G$ -action:*

Orbit symbol	Symmetry type
$\mathbf{D}_n(2r, \ell p)$	$\mathbf{D}_{2n}(r, \ell p)$
$\mathbb{T}(ke, 2\ell v)$	$\mathbb{O}(kv, \ell f)$

where  $k, \ell = 0, 1$ .

2. For the  $\widehat{G}$ -action:

<i>Orbit symbol</i>	<i>Symmetry type</i>	<i>Orbit symbol</i>	<i>Symmetry type</i>
$\mathbf{C}_n(R, \ell p)$	$\mathbf{C}_{nv}(R, \ell p)$	$\mathbf{C}_{nh}(R^e, 2\ell p)$	$\mathbf{D}_{nh}(r, \ell p)$
$\mathbf{C}_{nh}(R, 2\ell p)$	$\mathbf{D}_{nh}(R, \ell p)$	$\mathbf{D}_n(2r, \ell p)$	$\mathbf{D}_{2nh}(r, \ell p)$
$\mathbf{D}_{nh}(r, r', \ell p)$	$\mathbf{D}_{2nh}(r, \ell p)$	$\mathbf{D}_{nd}(r, \ell p)$	$\mathbf{D}_{2nh}(r, \ell p)$
$\mathbf{C}_i(2R)$	$\mathbf{D}_{2h}(R)$	$\mathbf{S}_{2n}(R)$	$\mathbf{D}_{nd}(R)$
$\mathbb{T}(v)$	$\mathbb{T}_d(v)$	$\mathbb{T}(ke, 2\ell v)$	$\mathbb{O}_h(kv, \ell f)$
$\mathbb{T}_d(ke, 2v)$	$\mathbb{O}_h(kv, f)$	$\mathbb{T}_h(ke, \ell v)$	$\mathbb{O}_h(kv, \ell f)$
$\mathbb{T}_d(e)$	$\mathbb{O}_h(v)$		
$\mathbb{O}(k_v v, k_e e, k_f f)$	$\mathbb{O}_h(k_v v, k_e e, k_f f)$	$\mathbb{I}(k_v v, k_e e, k_f f)$	$\mathbb{I}_h(k_v v, k_e e, k_f f)$

where  $k = 0, 1$  and  $\ell = 0, 1$  or  $2$  if appropriate.

If the vortices are not all identical, then some of these point-orbit symbols will be isotropy types. For example  $\mathbf{C}_i(2R)$  is an isotropy type if  $\Lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2)$  with  $\lambda_1 \neq \lambda_2$ .

PROOF This follows from an exhaustive inspection of Tables 1 and 2.  $\square$

### 1.3.1 Topology of strata

Having described which strata exist for which values of  $N$ , we now proceed to consider their basic topological properties, *viz.* dimension and connectedness. The fact that the  $G$  action is symplectic implies that all the fixed point spaces for the  $G$ -action are even-dimensional, and one can deduce that the strata are all connected. The situation is more complicated for the  $\widehat{G}$  action, but it turns out (Proposition 1.8) that provided all the vortices are identical, then all strata in the orbit space (quotient space)  $\mathcal{P}/\widehat{G}$  are connected. In §1.3.2 below we describe how the different strata “fit together”.

**Proposition 1.7** *Let  $\Sigma < G = \mathbf{SO}(3) \times S(\Lambda)$  be any isotropy subgroup for the  $G$ -action on  $\mathcal{P}$ , then the corresponding orbit type set in the orbit space  $\mathcal{P}/G$  is connected. Furthermore, if  $\Sigma$  is of type  $\Gamma(k_1 a_1, \dots, k_r a_r, k_R R)$ , then  $\dim \text{Fix}(\Sigma) = 2k_R$ . Here the  $a_i$  represent the non-regular point-orbits: that is, those of type  $r, p, v, f, e$ , while  $R$  denotes the unique regular point-orbit type. Furthermore the corresponding stratum in the orbit space  $\mathcal{P}/G$  satisfies*

$$\dim(\mathcal{P}/G)_{(\Sigma)} = \dim \text{Fix}(\Sigma, \mathcal{P}) - \dim(N(\Gamma)),$$

where  $N(\Gamma)$  is the normalizer of  $\Gamma = \pi(\Sigma)$  in  $\mathbf{SO}(3)$ .

Note that the subgroups  $\mathbf{C}_n$  and  $\mathbf{D}_n$  have 1-dimensional normalizers, while the cubic groups have 0-dimensional normalizers.

PROOF One can proceed by induction on the number of point-orbits  $\mathcal{O}$  of  $\Gamma$ , since if  $\Sigma < G$  is of type  $\Gamma(k_1 \mathcal{O}_1, \dots, k_r \mathcal{O}_r)$  and  $\Sigma'$  of type  $\Gamma(k_1 \mathcal{O}_1, \dots, k_r \mathcal{O}_r, \mathcal{O})$  (the same  $\Gamma$ ), then

$$\text{Fix}(\Sigma', \mathcal{P}') = \left( \text{Fix}(\Sigma, \mathcal{P}) \times \text{Fix}(\Sigma_{\mathcal{O}}, \mathcal{P}_{\mathcal{O}}) \right) \setminus \Delta$$

where  $\Sigma_{\mathcal{O}} = \Gamma(\mathcal{O})$  acts on  $\mathcal{P}_{\mathcal{O}} \subset (S^2)^{|\mathcal{O}|}$ , and  $\Delta$  consists of any diagonal points in this Cartesian product, and  $\mathcal{P}'$  is  $\mathcal{P} \times \mathcal{P}_{\mathcal{O}} \setminus \Delta$ .

It is therefore sufficient to establish the result for individual point-orbits, and this is clear by inspection. For example, for  $\Gamma = \mathbf{C}_n$ , the fixed point set for  $\mathbf{C}_n(R)$  is connected, while that for  $\mathbf{C}_n(p)$  is not: it has two components, one for each pole. Thus, for  $\mathbf{C}_n(kR, \ell p)$  the fixed point set has at most two components and these are identified by rotating the sphere so that the poles are exchanged, and possibly relabelling the vortices (for  $\ell = 2$ , two of the four possible components are excluded as they involve coincident vortices). Consequently, these components are identified by a symmetry operation, which means that they give rise to the same set in the orbit space  $\mathcal{P}/G$ .

Finally, note that the set of points with a given isotropy subgroup  $G_x$  does not in general coincide with the fixed point set for that subgroup as there is a submanifold of the fixed point set with higher isotropy. However, since all the fixed point spaces are symplectic, this submanifold is of codimension at least 2, so its complement is still connected. A similar argument shows that removing  $\Delta$  does not disconnect the set either.

For the dimension in the orbit space, it is a fact about actions of compact Lie groups, that the image of  $\text{Fix}(\Sigma)$  is of dimension  $\dim \text{Fix}(\Sigma) - \dim(N(\Sigma)/\Sigma)$ . In this case, we have that  $N_G(\Sigma) = N_{\mathbf{SO}(3)}(\Gamma) \times N_{S(\Lambda)}(\pi_2(\Sigma))$ , where  $\pi_2 : G \rightarrow S(\Lambda)$  is the Cartesian projection. Since both  $\Sigma$  and  $S(\Lambda)$  are finite, the result follows.  $\square$

We now turn to the case of the  $\widehat{G}$  action. The sets of points in the orbit space  $\mathcal{P}/\widehat{G}$  with a given  $\widehat{G}$ -orbit type is not necessarily connected in general as the following two examples show.

Firstly, suppose there are  $N$  vortices all with distinct vorticities and consider the configurations where all  $N$  lie on the equator, that is orbit type  $\mathbf{C}_h(NE)$ . There are  $(N-1)!/2$  connected components with this symmetry type in  $\mathcal{P}/\widehat{G}$  (here  $\widehat{G} = \mathbf{O}(3)$ ).

A second example, which even occurs when the vortices are identical, is given by the two types  $\mathbf{C}_{nv}(2R)$  and  $\mathbf{C}_{nv}(R, R')$  — see Remark 1.4. If  $n$  is odd these are two connected components of the same fixed point set. It should be noted that the two strata in question are geometrically distinct, and cannot be identified by elements of  $\mathbf{O}(3) \times S_N$ .

To avoid the first of the two problems, we restrict attention to the case where the vortices are all identical, and to avoid the second we consider point-orbit types, rather than orbit types.

**Proposition 1.8** *Suppose all the vortices are identical. Let  $\Sigma < \mathbf{O}(3) \times S_N$  be an isotropy subgroup, and  $\Gamma(k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d)$  a corresponding point-orbit type. Then that point-orbit type set is connected in  $\mathcal{P}/\widehat{G}$ , and is of dimension*

$$\sum_{j=1}^d k_j \dim(\mathcal{O}_j) - \dim(N(\Gamma)),$$

where  $N(\Gamma)$  is the normalizer of  $\Gamma$  in  $\widehat{G}$ , and  $\dim(\mathcal{O}_j)$  is the dimension as given in the fifth column of Table 2.

**PROOF** Observe that the point-orbits  $k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d$  correspond to a decomposition of the set  $\mathcal{N} = \{1, 2, \dots, N\}$  into disjoint subsets  $\mathcal{N}_1 \cup \dots \cup \mathcal{N}_d$  where  $\mathcal{N}_j$  consists of the labels of all the points in the  $k_j$  orbits of type  $\mathcal{O}_j$ . Write  $N_j = |\mathcal{N}_j|$ . Let  $S(\mathcal{N}_j)$  be the subgroup of  $S(\Lambda)$  of permutations of these  $N_j$  points. Then the homomorphism  $\psi : \Gamma \rightarrow S(\Lambda)$  which determines  $\Sigma$  satisfies  $\text{Im}(\psi) < S(\mathcal{N}_1) \times \dots \times S(\mathcal{N}_d)$ . Thus,  $\psi$  can be written as  $\psi = (\psi_1, \dots, \psi_d)$ , with  $\psi_j : \Gamma \rightarrow S(\mathcal{N}_j)$ . For  $j = 1, \dots, d$  write

$$\mathcal{P}_j = \left( \prod_{i=1}^{N_j} S^2 \right) \setminus \Delta.$$

$\mathcal{P}_j$  is the factor of the phase space corresponding to the points in  $\mathcal{N}_j$ , so that  $\mathcal{P} = \left(\prod_{j=1}^d \mathcal{P}_j\right) \setminus \Delta$ , and  $S(\mathcal{N}_j)$  acts on  $\mathcal{P}_j$ . Also let  $\Sigma_j$  be the graph of  $\psi_j$ .

Now observe that

$$\text{Fix}(\Sigma, \mathcal{P}) = \text{Fix}(\Sigma_1, \mathcal{P}_1) \times \cdots \times \text{Fix}(\Sigma_d, \mathcal{P}_d)$$

(there is no need to remove diagonal points as different types of point-orbit cannot occupy the same points on the sphere), and furthermore

$$\frac{\text{Fix}(\Sigma, \mathcal{P})}{N_2(\Sigma)} = \frac{\text{Fix}(\Sigma_1, \mathcal{P}_1)}{N_2(\Sigma_1)} \times \cdots \times \frac{\text{Fix}(\Sigma_d, \mathcal{P}_d)}{N_2(\Sigma_d)},$$

where  $N_2(\Sigma)$  is the normalizer of the image  $\psi(\Gamma)$  in  $S(\Lambda) \simeq S_N$ , and similarly  $N_2(\Sigma_j)$  is the normalizer of the image  $\psi_j(\Gamma)$  in  $S(\mathcal{N}_j) \simeq S_{N_j}$ .

We wish to show then that for each  $j = 1, \dots, d$ , the set  $\text{Fix}(\Sigma_j, \mathcal{P}_j)/N_2(\Sigma_j)$  is connected, and of dimension  $k_j \dim(\mathcal{O}_j)$ . The result then follows. But this follows from an inspection of Table 2. For example, for  $\mathbf{C}_{nv}(kR)$  the fixed point set is parametrized by specifying  $k$  distinct points on the interval  $(-\pi/2, \pi/2)$  (the latitude of each ring), and this set has  $k!$  connected components. However, after factoring out the permutation group  $N_2(\Sigma)$  these are all identified, leaving one component in the orbit space.

The remainder of the proof follows the last paragraph of the proof of Proposition 1.7.  $\square$

### 1.3.2 Adjacencies

Recall that the set of points with a given orbit type is called a stratum of the orbit type stratification. The orbit types, or the strata, are partially ordered by conjugacy of one isotropy subgroup to a subgroup of another or, equivalently, inclusion of the smaller orbit type stratum (corresponding to the larger isotropy subgroup) into the closure of the larger orbit type stratum. In this case we say that the larger stratum *specializes* to the smaller stratum, or that the smaller is *adjacent* to the larger. The resulting partially ordered set is called the *lattice* of orbit types. A good general reference for this lattice is [10].

We wish to give necessary and sufficient conditions for one stratum to be adjacent to another. It is a standard result on the actions of Lie groups that if  $\mathcal{S}$  is adjacent to  $\mathcal{S}'$  then the isotropy subgroup corresponding to  $\mathcal{S}'$  is *subconjugate* to the isotropy subgroup of  $\mathcal{S}$ . The following lemma gives a useful geometric criterion for when one isotropy subgroup is subconjugate to another.

**Lemma 1.9** *Let  $\Sigma$  and  $\Sigma'$  be two isotropy subgroups of  $G$  or  $\widehat{G}$ , of orbit types  $\Gamma(k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d)$  and  $\Gamma'(k'_1\mathcal{O}'_1, \dots, k'_e\mathcal{O}'_e)$  respectively. Then  $\Sigma'$  is subconjugate to  $\Sigma$  if and only if  $\Gamma' < \Gamma$  and the restriction to  $\Gamma'$  of the action of  $\Gamma$  on  $k_1\mathcal{O}_1 \cup \dots \cup k_d\mathcal{O}_d$  is isomorphic (as a  $\Gamma'$ -set) to the action of  $\Gamma'$  on  $k'_1\mathcal{O}'_1 \cup \dots \cup k'_e\mathcal{O}'_e$ .*

**PROOF** Recall that  $\Sigma$  is the graph of a homomorphism  $\psi : \Gamma \rightarrow S(\Lambda)$ , and similarly  $\Sigma'$  the graph of  $\psi' : \Gamma' \rightarrow S(\Lambda)$ . Clearly then,  $\Sigma'$  is a subgroup of  $\Sigma$  if and only if  $\Gamma' < \Gamma$  and  $\psi'$  is the restriction of  $\psi$  to  $\Gamma'$ . The homomorphism  $\psi$  expresses how each element of  $\Gamma$  permutes the vortices, so the restriction of  $\psi$  to  $\Gamma'$  is just the permutation action of  $\Gamma'$  on the set  $k_1\mathcal{O}_1 \cup \dots \cup k_e\mathcal{O}_e$ , as required.  $\square$

We wish to apply this group theoretic result to the adjacencies of strata in the two orbit type stratifications. Suppose  $\Sigma$  is an isotropy subgroup of point-orbit type  $\Gamma(k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d)$ , and let  $\Gamma' < \Gamma$ . Then the action of  $\Gamma'$  on the set  $k_1\mathcal{O}_1 \cup \dots \cup k_d\mathcal{O}_d$  decomposes into orbits  $k'_1\mathcal{O}'_1 \cup \dots \cup k'_e\mathcal{O}'_e$ , so that the corresponding action of  $\Sigma' = \pi^{-1}(\Gamma')$ , where  $\pi : \Sigma \rightarrow \mathbf{O}(3)$ , is of type  $\Gamma'(k'_1\mathcal{O}'_1, \dots, k'_e\mathcal{O}'_e)$ . We call this latter type the *restricted point-orbit type*.

It follows from this lemma and the standard result from Lie group actions mentioned above, that if a stratum  $\mathcal{S}$  of type  $\Gamma(k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d)$  is adjacent to a stratum  $\mathcal{S}'$  of type  $\Gamma'(k'_1\mathcal{O}'_1, \dots, k'_e\mathcal{O}'_e)$ , then  $\Gamma'$  acts

on  $k'_1\mathcal{O}'_1 \cup \dots \cup k'_e\mathcal{O}'_e$  as the restriction to  $\Gamma'$  of the  $\Gamma$  action on the set  $k_1\mathcal{O}_1 \cup \dots \cup k_d\mathcal{O}_d$ . We now state the converse for systems of *identical* point vortices.

**Theorem 1.10** *Suppose there are  $N > 2$  identical point vortices. Consider the point-orbit type stratum  $\mathcal{S}$  of type  $\Gamma(k_1\mathcal{O}_1, \dots, k_r\mathcal{O}_r)$ , and let  $\Gamma' < \Gamma$  be such that the restricted point-orbit type  $\Gamma'(k'_1\mathcal{O}'_1, \dots, k'_r\mathcal{O}'_r)$  is also an isotropy type. Then the corresponding stratum  $\mathcal{S}'$  is adjacent to  $\mathcal{S}$ .*

**PROOF** Let  $\Sigma$  be a subgroup of  $\widehat{G}$  corresponding to  $\mathcal{S}$ , and  $\Sigma'$  corresponding to  $\mathcal{S}'$ . Since  $\Sigma' < \Sigma$ , we have that

$$\text{Fix}(\Sigma, \mathcal{P}) \subset \text{Fix}(\Sigma', \mathcal{P}).$$

By Proposition 1.7 the images in  $\mathcal{P}/\widehat{G}$  of these two fixed-point sets are connected, and so equal to the closures  $\overline{\mathcal{S}}$  and  $\overline{\mathcal{S}'}$ . Consequently,

$$\mathcal{S}' \subset \overline{\mathcal{S}'} \subset \overline{\mathcal{S}},$$

as required.  $\square$

**Remark 1.11** In fact the result given in the theorem remains valid even if the vortices are not identical, at least for the case of  $G$  orbit type strata. The proof is identical, using Proposition 1.7 in place of Proposition 1.8. It appears that the same is true for the  $\widehat{G}$ -action, though we do not have a proof.  $\diamond$

### 1.3.3 Orbit type lattices and minimal strata

Using the results on the dimensions and adjacencies of the strata, it is now a fairly routine exercise to give complete point-orbit type lattices of  $N$  identical vortices. The complexity of such lattices increases quickly with  $N$ , so in Figures 1 and 2 we show some examples of these lattices, for  $N$  up to 6 for  $G$  and up to 5 for  $\widehat{G}$ .

A stratum is said to be *minimal* if it is minimal with respect to the partial ordering described above, i.e. it does not specialize to another stratum, or in other words, it is closed in  $\mathcal{P}$  (or in  $\mathcal{P}/\widehat{G}$ ). In Section 2, and in particular Theorem 2.2, we show that every minimal stratum contains an equilibrium.

Table 3 lists the  $G$ -orbit type strata which can be minimal for the case of  $N$  identical vortices and gives the values of  $N$  for which they are minimal.

Stratum	$N$ for minimality	Dimension in quotient
$\mathbf{C}_n(R, p)$ $n \geq 4$	$N = n + 1$	1
$\mathbf{D}_n(r)$ $n \neq 2$	$N = n$	0
$\mathbf{D}_n(r, p)$ $n \neq 2, 4$	$N = n + 2$	0
$\mathbb{T}(k_R R, k_e e, v)$	$N/2 \equiv 2 \pmod{3}$	$2k_R$
$\mathbb{O}(k_R R, k_e e, k_f f, k_v v)$	$N/2 \equiv 0, 1 \pmod{3}$	$2k_R$
$\mathbb{I}(k_R R, k_e e, k_f f, k_v v)$	$N/2 \equiv 0, 1, 6, 10 \pmod{15}$	$2k_R$

Table 3: Minimal strata for the action of  $G = \mathbf{SO}(3) \times S_N$  on configurations of  $N$  identical vortices.

**Proposition 1.12** *The minimal strata for the action of  $\widehat{G} = \mathbf{O}(3) \times S_N$  on the phase space  $\mathcal{P}$  of  $N > 2$  identical vortices are as follows.*

1. Strata with isotropy subgroups  $\mathbb{I}_h$  and  $\mathbb{O}_h$  are minimal whenever they are non-empty.

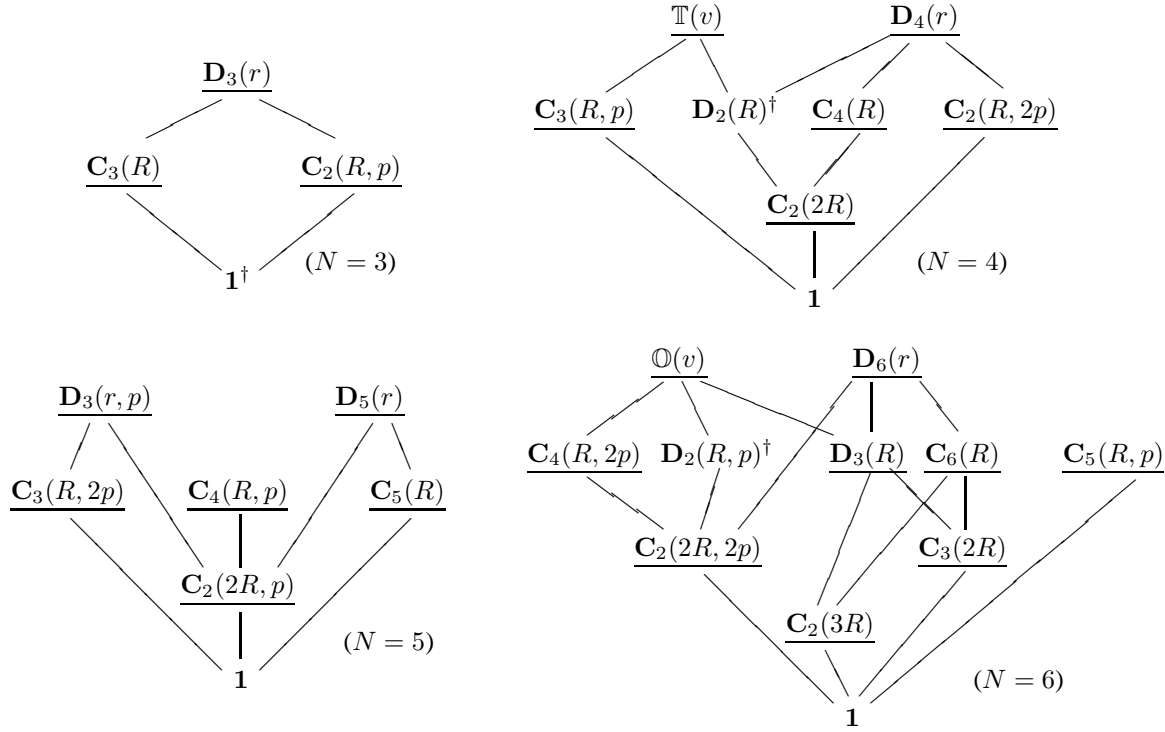


Figure 1: The orbit type lattice for the action of  $\mathrm{SO}(3) \times S_N$  for  $N = 3, 4, 5$  and  $6$  identical vortices. The underlined strata are those which we prove contain relative equilibria for any  $G$ -invariant Hamiltonian, and those marked with a dagger  $\dagger$  do not contain any for the point-vortex Hamiltonian. See 4.

2. A stratum  $\mathbb{T}_d(k_R, k_E, k_e, k_v)$  is minimal if and only if  $k_v = 1$ .
3. For  $n \neq 2, 4$  a stratum  $\mathbf{D}_{nh}(k_{R_s}, k_{R_s^e}, k_R, k_{R'}, k_r, k_{r'}, k_p)$  is minimal if and only if both the following conditions hold:

- (a)  $k_R \neq k_{R'}$  or  $k_r \neq k_{r'}$ ;
- (b) There do not exist an odd prime  $k$  and non-negative integers  $a, b$  such that

$$k_{R_s} = \frac{(k-1)}{2} (k_R + k_{R'}) + ka$$

$$k_{R_s^e} = \frac{(k-1)}{2} (k_r + k_{r'}) + kb.$$

4. A stratum  $\mathbf{D}_{4h}(k_{R_s}, k_{R_s^e}, k_R, k_{R'}, k_r, k_{r'}, k_p)$  is minimal if in addition to the conditions of the previous item it also satisfies the following pair of conditions:

- (a)  $k_r \neq k_p$  or  $k_R \neq k_{R'} + 2k_{R_s^e}$  or  $k_{R'} < k_{R_s} - 3 \lfloor k_{R_s}/3 \rfloor$ ;
- (b)  $k_{r'} \neq k_p$  or  $k_{R'} \neq k_r + 2k_{R_s^e}$  or  $k_R < k_{R_s} - 3 \lfloor k_{R_s}/3 \rfloor$ .

where  $\lfloor x \rfloor$  is the integer part of  $x$ .

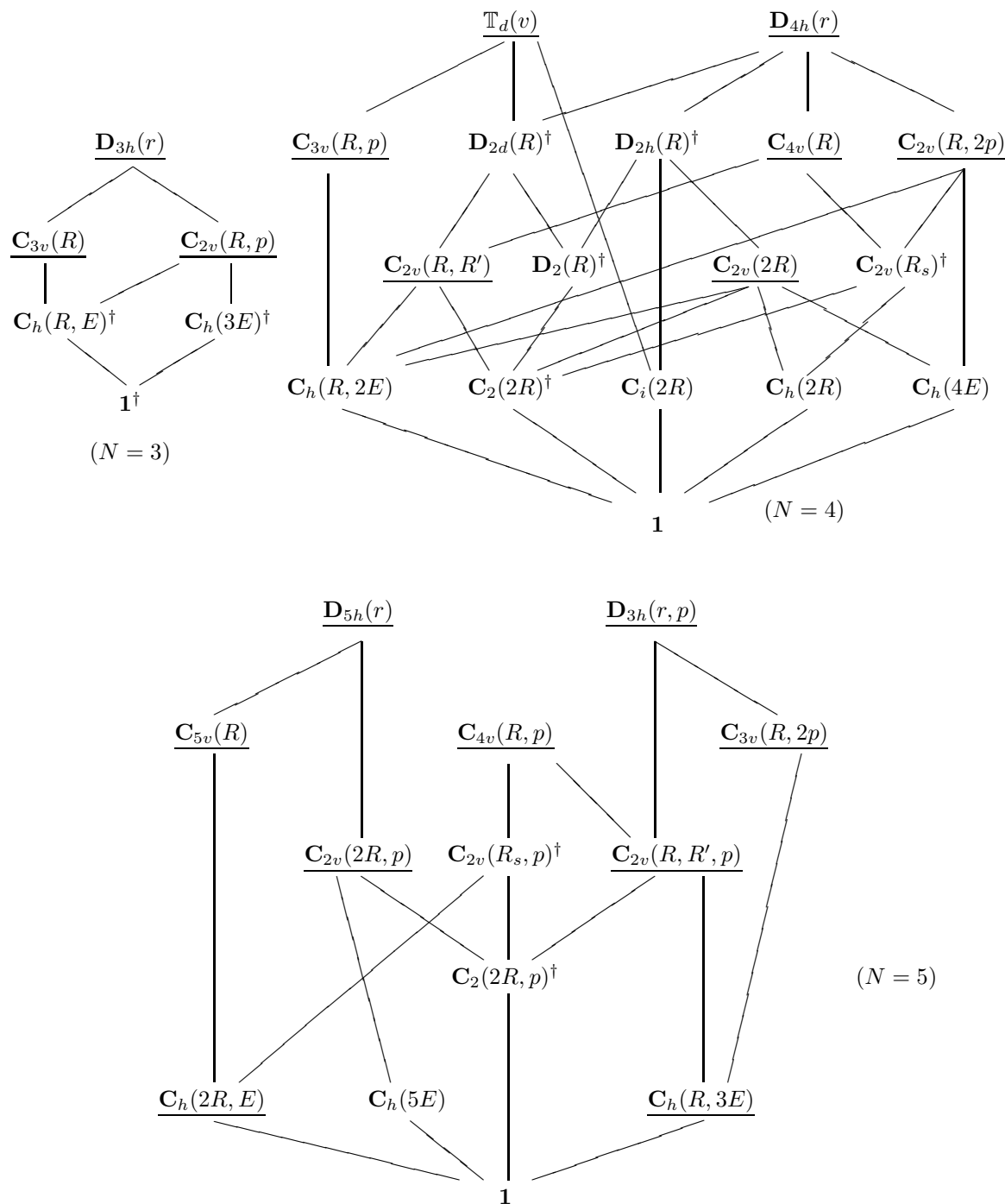


Figure 2: Orbit type lattice for the action of  $\mathbf{O}(3) \times S_N$  for  $N = 3, 4$  and  $5$  identical vortices. The underlined strata are those which we prove contain relative equilibria for any  $\widehat{G}$ -invariant Hamiltonian, and those marked with a dagger  $^\dagger$  do not contain any for the point-vortex Hamiltonian. See 4.

5. For  $n \geq 4$  a stratum  $\mathbf{C}_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$  is minimal if and only if  $k_p = 1$ ,  $k_R \neq k_{R'}$  and there do not exist an odd prime  $k$  and non-negative integer  $a$  such that

$$k_{R_s} = \frac{(k-1)}{2}(k_R + k_{R'}) + ka.$$

6. A stratum  $\mathbf{C}_{3v}(k_{R_s}, k_R, k_{R'}, k_p)$  is minimal if and only if in addition to the conditions of the previous item it also does not satisfy

$$k_{R_s} = \frac{1}{2}(k_R - k_{R'} - k_p) + 4a$$

for any non-negative integer  $a$ .

7. A stratum  $\mathbf{C}_{2v}(k_{R_s}, k_R, k_{R'}, k_p)$  is minimal if and only if in addition to the conditions of item 5 there do not exist an odd integer  $k \geq 3$  and non-negative integers  $a$  and  $b$  such that both the following conditions hold:

(a)  $k_{R_s} = \frac{(k-1)}{2}k_R + ka$  or  $k_{R_s} = \frac{(k-1)}{2}(k_R - 1) + ka$ ;

(b)  $k_{R'} = \frac{(k-1)}{2}k_p + kb$ .

8. Strata with isotropy subgroups conjugate to any other finite subgroups of  $\widehat{G}$  are not minimal.

The proof of the proposition is a routine, but extremely tedious, case-by-case analysis which makes heavy use of Lemma 1.9 and Theorem 1.10. The following result is an easy consequence of the proposition.

**Corollary 1.13** For a system of  $N$  identical vortices there exists a minimal  $\widehat{G}$  orbit type stratum with isotropy subgroups projecting to:

1.  $\mathbb{I}_h$  if and only if  $N$  is even and  $N/2 \equiv 0, 1, 6, 10 \pmod{15}$ ;
2.  $\mathbb{O}_h$  if and only if  $N$  is even and  $N/2 \equiv 0, 1 \pmod{3}$ ;
3.  $\mathbb{T}_d$  if and only if  $N$  is even and  $N/2 \equiv 2 \pmod{3}$ ;
4.  $\mathbf{D}_{nh}$  with  $n \neq 2, 4$  if and only if  $N \equiv 0, 2 \pmod{n}$ ;
5.  $\mathbf{D}_{4h}$  if  $N$  is even and  $N/2 \equiv 2 \pmod{3}$  (but also for some other  $N$ );
6.  $\mathbf{C}_{nv}$  with  $n \geq 3$  if and only if  $N \equiv 1 \pmod{n}$ ;
7.  $\mathbf{C}_{2v}$  if and only if  $N$  is odd and  $N \geq 9$ .

Table 4 lists the minimal strata for  $N = 3 \dots 12$  identical vortices for both the  $G$  and  $\widehat{G}$  actions.

## 1.4 Orbit types for fixed centres of vorticity

In this subsection we describe the orbit types that can occur in a single fibre  $\Phi^{-1}(\mu)$  or, equivalently, the symmetry types of configurations of vortices with fixed centre of vorticity  $\mu$ . The actions of  $G$  and  $\widehat{G}$  on  $\mathcal{P}$  restrict, respectively, to actions of the momentum isotropy subgroups  $G_\mu$  and  $\widehat{G}_\mu$  on  $\Phi^{-1}(\mu)$ . If  $\mu = 0$  then  $G_\mu = G$  and  $\widehat{G}_\mu = \widehat{G}$ . When  $\mu \neq 0$  then without loss of generality we may suppose  $G_\mu = \mathbf{SO}(2) \times S_\Lambda$  and  $\widehat{G}_\mu = \mathbf{O}(2) \times S_\Lambda$ . A stratum is called a *zero-momentum stratum* if it is contained in  $\Phi^{-1}(0)$ .

N	Minimal $G$ -strata	Minimal $\widehat{G}$ -strata	Minimal $\widehat{G}$ -strata in $\Phi^{-1}(\mu)$ for $\mu \neq 0$
3	$\mathbf{D}_3(r)$	$\mathbf{D}_{3h}(r)$	$\mathbf{C}_{3v}(R), \mathbf{C}_{2v}(R, p)$
4	$\mathbf{D}_4(r)$ $\mathbb{T}(v)$	$\mathbf{D}_{4h}(r)$ $\mathbb{T}_d(v)$	$\mathbf{C}_{4v}(R), \mathbf{C}_{2v}(2R), \mathbf{C}_{3v}(R, p)$ $\mathbf{C}_{2v}(R, 2p), \mathbf{C}_{2v}(2R)$
5	$\mathbf{C}_4(R, p)$ $\mathbf{D}_5(r), \mathbf{D}_3(r, p)$	$\mathbf{C}_{4v}(R, p)$ $\mathbf{D}_{5h}(r), \mathbf{D}_{3h}(r, p)$	$\mathbf{C}_{5v}(R), \mathbf{C}_{4v}(R, p), \mathbf{C}_{2v}(2R, p)$ $\mathbf{C}_{3v}(R, 2p)$
6	$\mathbf{C}_5(R, p)$ $\mathbf{D}_6(r)$ $\mathbb{O}(v)$	$\mathbf{C}_{5v}(R, p)$ $\mathbf{D}_{6h}(r), \mathbf{D}_{3h}(R)$ $\mathbb{O}_h(v)$	$\mathbf{C}_{6v}(R), \mathbf{C}_{3v}(2R), \mathbf{C}_{2v}(3R)$ $\mathbf{C}_{2v}(2R, R')$ $\mathbf{C}_{5v}(R, p), \mathbf{C}_{4v}(R, 2p), \mathbf{C}_{2v}(2R, 2p)$
7	$\mathbf{C}_6(R, p)$ $\mathbf{D}_7(r), \mathbf{D}_5(r, p)$	$\mathbf{C}_{6v}(R, p), \mathbf{C}_{3v}(2R, p)$ $\mathbf{D}_{7h}(r), \mathbf{D}_{5h}(r, p)$	$\mathbf{C}_{7v}(R), \mathbf{C}_{6v}(R, p), \mathbf{C}_{3v}(2R, p)$ $\mathbf{C}_{2v}(3R, p), \mathbf{C}_{2v}(2R, R', p), \mathbf{C}_{5v}(R, 2p)$
8	$\mathbf{C}_7(R, p)$ $\mathbf{D}_8(r), \mathbf{D}_6(r, p)$ $\mathbb{O}(f)$	$\mathbf{C}_{7v}(R, p)$ $\mathbf{D}_{8h}(r), \mathbf{D}_{6h}(r, p), \mathbf{D}_{3h}(R, p)$ $\mathbb{O}_h(f)$	$\mathbf{C}_{8v}(R), \mathbf{C}_{4v}(2R), \mathbf{C}_{2v}(4R)$ $\mathbf{C}_{2v}(3R, R'), \mathbf{C}_{2v}(R_s, 2R), \mathbf{C}_{7v}(R, p)$ $\mathbf{C}_{6v}(R, 2p), \mathbf{C}_{3v}(2R, 2p), \mathbf{C}_{2v}(3R, 2p)$ $\mathbf{C}_{2v}(2R, R', p)$
9	$\mathbf{C}_8(R, p)$ $\mathbf{D}_9(r), \mathbf{D}_7(r, p)$	$\mathbf{C}_{8v}(R, p), \mathbf{C}_{4v}(2R, p),$ $\mathbf{C}_{2v}(R_s, 2R, p), \mathbf{D}_{9h}(r), \mathbf{D}_{7h}(r, p),$ $\mathbf{D}_{3h}(R, r), \mathbf{D}_{3h}(R, r')$	$\mathbf{C}_{9v}(R), \mathbf{C}_{3v}(3R), \mathbf{C}_{3v}(2R, R')$ $\mathbf{C}_{8v}(R, p), \mathbf{C}_{4v}(2R, p), \mathbf{C}_{2v}(4R, p)$ $\mathbf{C}_{2v}(3R, R', p), \mathbf{C}_{2v}(R_s, 2R, p), \mathbf{C}_{7v}(R, 2p)$
10	$\mathbf{C}_9(R, p)$ $\mathbf{D}_{10}(r), \mathbf{D}_8(r, p)$ $\mathbb{T}(e, v)$	$\mathbf{C}_{9v}(R, p), \mathbf{C}_{3v}(3R, p)$ $\mathbf{D}_{10h}(r), \mathbf{D}_{8h}(r, p),$ $\mathbf{D}_{5h}(R), \mathbf{D}_{4h}(R, p)$ $\mathbb{T}_d(e, v)$	$\mathbf{C}_{10v}(R), \mathbf{C}_{5v}(2R), \mathbf{C}_{2v}(5R)$ $\mathbf{C}_{2v}(4R, R'), \mathbf{C}_{2v}(3R, 2R'), \mathbf{C}_{2v}(R_s, 3R)$ $\mathbf{C}_{2v}(R_s, 2R, R'), \mathbf{C}_{9v}(R, p), \mathbf{C}_{3v}(3R, p)$ $\mathbf{C}_{3v}(2R, R', p), \mathbf{C}_{8v}(R, 2p), \mathbf{C}_{4v}(2R, 2p)$ $\mathbf{C}_{2v}(4R, 2p), \mathbf{C}_{2v}(3R, R', 2p),$ $\mathbf{C}_{2v}(R_s, 2R, 2p)$
11	$\mathbf{C}_{10}(R, p)$  $\mathbf{D}_{11}(r), \mathbf{D}_9(r, p)$	$\mathbf{C}_{10v}(R, p), \mathbf{C}_{5v}(2R, p),$ $\mathbf{C}_{2v}(R_s, 3R, p)$ $\mathbf{C}_{2v}(3R, 2R', p)$ $\mathbf{D}_{11h}(r), \mathbf{D}_{9h}(r, p), \mathbf{D}_{3h}(R, r, p)$ $\mathbf{D}_{3h}(R, r', p)$	$\mathbf{C}_{11v}(R), \mathbf{C}_{10v}(R, p), \mathbf{C}_{5v}(2R, p)$  $\mathbf{C}_{2v}(5R, p), \mathbf{C}_{2v}(4R, R', p), \mathbf{C}_{2v}(3R, 2R', p)$ $\mathbf{C}_{2v}(R_s, 3R, p), \mathbf{C}_{2v}(R_s, 2R, R', p)$ $\mathbf{C}_{9v}(R, 2p), \mathbf{C}_{3v}(3R, 2p), \mathbf{C}_{3v}(2R, R', 2p)$
12	$\mathbf{C}_{11}(R, p)$ $\mathbf{D}_{12}(r), \mathbf{D}_{10}(r, p)$  $\mathbb{O}(e)$ $\mathbb{I}(v)$	$\mathbf{C}_{11v}(R, p)$ $\mathbf{D}_{12h}(r), \mathbf{D}_{10h}(r, p), \mathbf{D}_{6h}(R),$ $\mathbf{D}_{5h}(R, p), \mathbf{D}_{4h}(R, r), \mathbf{D}_{4h}(R, r'),$ $\mathbf{D}_{3h}(R, r, r'),$ $\mathbb{O}_h(e)$ $\mathbb{I}_h(v)$	$\mathbf{C}_{12v}(R), \mathbf{C}_{6v}(2R), \mathbf{C}_{4v}(3R)$ $\mathbf{C}_{4v}(2R, R'), \mathbf{C}_{3v}(4R), \mathbf{C}_{3v}(3R, R')$ $\mathbf{C}_{3v}(R_s, 2R), \mathbf{C}_{2v}(6R), \mathbf{C}_{2v}(5R, R')$ $\mathbf{C}_{2v}(4R, 2R'), \mathbf{C}_{2v}(R_s, 4R),$ $\mathbf{C}_{2v}(R_s, 3R, R')$ $\mathbf{C}_{11v}(R, p), \mathbf{C}_{10v}(R, 2p), \mathbf{C}_{5v}(2R, 2p)$ $\mathbf{C}_{2v}(5R, 2p), \mathbf{C}_{2v}(4R, R', 2p),$ $\mathbf{C}_{2v}(3R, 2R', 2p)$ $\mathbf{C}_{2v}(R_s, 3R), \mathbf{C}_{2v}(R_s, 2R, R', 2p)$

Table 4: Minimal strata for the actions of  $G = \mathbf{SO}(3) \times S_N$  and  $\widehat{G} = \mathbf{O}(3) \times S_N$  on configurations of  $N$  identical vortices with  $3 \leq N \leq 12$ .

**Proposition 1.14** *All strata except those of types  $C_n$ ,  $C_{nv}$ ,  $C_h$  and  $1$  are zero-momentum strata.*

PROOF By the equivariance of  $\Phi$ , if  $\mathbf{x} \in \text{Fix}(\Sigma, \mathcal{P})$  then  $\Phi(\mathbf{x}) \in \text{Fix}(\Sigma, \mathbf{R}^3)$ . For all the symmetry groups listed previously, except the three mentioned in the proposition, we have  $\text{Fix}(\Sigma, \mathbf{R}^3) = 0$ . To see that each of the four remaining strata are not contained in  $\Phi^{-1}(0)$ , it is enough to find an example of a configuration with that symmetry and with  $\Phi \neq 0$ . This is an easy exercise left to the reader.  $\square$

Thus the isotropy subgroups of points in  $\Phi^{-1}(\mu)$  are greatly restricted when  $\mu \neq 0$ . For any particular collection of vorticities  $\Lambda$  it is a relatively straightforward exercise to calculate the minimal strata in  $\Phi^{-1}(\mu)$  for  $\mu \neq 0$ . The following result treats the case of  $N$  identical vortices. Without loss of generality we may assume that they all have unit vorticity. In this case  $|\mu|$  ranges from 0 up to  $N$ .

**Proposition 1.15** *For  $N$  identical vortices with unit vorticity:*

1. *The intersection of the closure of a stratum  $C_n(k_R, k_p)$  or  $C_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$  with  $\Phi^{-1}(\mu)$  is non-empty if and only if one of the following conditions holds:*

- (a)  $k_p = 0$  or  $1$  and  $0 \leq |\mu| < N$ ;
- (b)  $k_p = 2$  and  $0 \leq |\mu| < N - 2$ .

*Every stratum  $C_h(k_R, k_E)$  has a non-empty intersection with  $\Phi^{-1}(\mu)$  for  $0 \leq |\mu| < N$ .*

2. *The intersections of the  $\widehat{G}$  strata  $C_n(k_R, k_p)$  and  $C_h(k_R, k_E)$  with  $\Phi^{-1}(\mu)$  are never minimal in  $\Phi^{-1}(\mu)$ .*
3. *If  $C_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$  has a non-empty intersection with  $\Phi^{-1}(\mu)$  for  $\mu \neq 0$  then it is minimal in  $\Phi^{-1}(\mu)$  if and only if  $k_R \neq k_{R'}$  and there do not exist an odd prime  $k$  and non-negative integer  $a$  such that*

$$k_{R_s} = ka + \frac{(k-1)}{2}(k_R + k_{R'}).$$

*In each case it is assumed that  $N$  has a value for which the stratum is non-empty in  $\mathcal{P}$ .*

The proof is straightforward. Lemma 1.9 is used to prove the minimality statements. The minimal strata in  $\Phi^{-1}(\mu)$ ,  $\mu \neq 0$ , for  $N = 3 \dots 12$  identical vortices are listed in Table 4.

## 2 Existence of Relative Equilibria

In this section we give a number of results which state the existence of equilibria and relative equilibria with particular symmetry types. To a large extent the results depend only on the symmetries of the model, not on any particular form of the Hamiltonian. However for some of them we do require that  $H(\mathbf{x}) \rightarrow \infty$  as  $\mathbf{x} \rightarrow \Delta$ , a property which is always satisfied if the vorticities all have the same sign, but which may fail if there are vorticities of opposite signs. For the bifurcation result in section 2.3 we also require a non-degeneracy condition to be satisfied.

Recall that a relative equilibrium is a trajectory that lies in the group orbit or, (almost) equivalently, an invariant group orbit. The fact that the trajectory lies in the group orbit means that the Hamiltonian vector field is always tangent to this orbit, so that  $\mathbf{x}$  lies on a relative equilibrium if and only if there is a  $\xi \in \mathfrak{so}(3)$  for which

$$X_H(\mathbf{x}) = \xi_{\mathcal{P}}(\mathbf{x}). \quad (2.1)$$

This value of  $\xi$  is the *angular velocity* of the relative equilibrium in question. Using the symplectic form this becomes  $dH_{\mathbf{x}} = \xi \cdot d\Phi_{\mathbf{x}}$  and so is equivalent to requiring  $\mathbf{x}$  to be a critical point of  $H_{\xi} = H - \xi \cdot \Phi$ . If the level set  $\Phi^{-1}(\mu)$  is non-singular, as it always is for point vortex systems if  $N > 2$ , then it follows that  $\mathbf{x} \in \Phi^{-1}(\mu)$  lies on a relative equilibrium if and only if  $\mathbf{x}$  is a critical point of the restriction of  $H$  to  $\Phi^{-1}(\mu)$ . Thus relative equilibria are given by constrained critical points of  $H$  in much the same way that equilibria are given by ordinary critical points.

If the point  $\mathbf{x}$  has a particular symmetry, then so must the angular velocity  $\xi$ , as the following result shows.

**Proposition 2.1** *Let  $\mathbf{x} \in \mathcal{P}$  satisfy (2.1), and let  $\widehat{G}_{\mathbf{x}}$  be the isotropy subgroup. Then  $\xi$  satisfies*

$$\text{Ad}_g \xi = \det(g)\xi$$

for all  $g \in \widehat{G}_{\mathbf{x}}$ , where  $\det(A, \sigma) = \det(A)$ .

PROOF If  $g \in \widehat{G}_{\mathbf{x}}$  then:

$$\xi \cdot d\Phi_{\mathbf{x}} = dH_{\mathbf{x}} = g^{-1} \circ dH_{\mathbf{x}} \circ g = g^{-1} \circ (\xi \cdot d\Phi_{\mathbf{x}}) \circ g = \text{Ad}_g \xi \cdot (g^{-1} \circ d\Phi_{\mathbf{x}} \circ g) = \det(g) \text{Ad}_g \xi \cdot d\Phi_{\mathbf{x}}$$

since  $d\Phi_{\mathbf{x}} \circ g = \det(g)g \circ d\Phi_{\mathbf{x}}$ . The result follows from this.  $\square$

We therefore define the action of  $\widehat{G}$  on  $\mathfrak{so}(3)$  by

$$g \cdot \xi = \det(g) \text{Ad}_g \xi.$$

This is evidently isomorphic to the action of  $\widehat{G}$  on  $\mathfrak{so}(3)^*$  given in Proposition 1.2.

## 2.1 Equilibria

Apart from its intrinsic interest, the following result serves as a prototype for the existence theorem for relative equilibria in §2.2 and the equilibrium points it provides are the starting points for the bifurcation theory in §2.3. Recall that an orbit type stratum is a *connected component* of the set of points with a given orbit type. It is therefore also a connected component of the set of points with a given point-orbit type — see Remark 1.4.

**Theorem 2.2** *Let  $X \subset \mathcal{P}$  be an orbit type stratum of the action of either  $G$  or  $\widehat{G}$  on  $\mathcal{P}$ .*

1. *If  $X$  consists of a single  $\mathbf{SO}(3)$  orbit then every point of  $X$  is an equilibrium configuration of every invariant Hamiltonian on  $\mathcal{P}$ .*
2. *If  $H(\mathbf{x}) \rightarrow \infty$  as  $\mathbf{x} \rightarrow \Delta \cap X$  in  $X$ , then there exists at least one equilibrium point on the closure of  $X$  in  $\mathcal{P}$ . In particular, if  $X$  is a minimal stratum then there exists at least one equilibrium point in  $X$ .*

PROOF The first statement follows immediately from the principle of symmetric criticality [34] since the orbits are necessarily critical orbits of  $H|_X$  and hence critical orbits of  $H$ .

For the second statement let  $X_0$  be a connected component of  $X \subset \mathcal{P} \subset (S^2)^N$  and let  $\overline{X}_0$  be the closure of  $X_0$  in  $\overline{\mathcal{P}} = (S^2)^N$ . Then  $\overline{X}_0$  is a compact submanifold of  $\overline{\mathcal{P}}$ . We claim that  $H|_{\overline{X}_0}$  has a minimum not in  $\Delta$ . This point is then an equilibrium point, for by the principle of symmetric criticality, a critical point of  $H|_{\overline{X}}$  is a critical point of  $H$ .

To prove the claim, note that if  $\overline{X}_0 \cap \Delta$  is empty, there is nothing more to prove since any continuous function on a compact set has a minimum. On the other hand if  $\overline{X}_0 \cap \Delta$  is non-empty, we need to use the fact that as  $\mathbf{x} \rightarrow \Delta$  so  $H(\mathbf{x}) \rightarrow \infty$ . Let  $\mathbf{x}_0 \in X_0$ . Then  $X_{\mathbf{x}_0} := \{\mathbf{x} \in \overline{X}_0 \mid H(\mathbf{x}) \leq H(\mathbf{x}_0)\}$  is a closed subset of  $\overline{X}_0$  and so compact, and satisfies  $X_{\mathbf{x}_0} \cap \Delta = \emptyset$ . It follows that  $H|_{\overline{X}_0}$  has a minimum in  $X_{\mathbf{x}_0}$ , which is contained in the closure of  $X_0$  in  $\mathcal{P}$ , as required.  $\square$

**Corollary 2.3** *For any  $\mathbf{O}(3) \times S_\Lambda$  invariant Hamiltonian on  $\mathcal{P}$  there exist equilibrium points in the strata:*

1.  $\mathbf{D}_{nh}(k_r, k_p)$  if  $\Lambda = (nk_r\lambda_r, 2k_p\lambda_p)$  and so  $N = nk_r + 2k_p$ ;
2.  $\mathbb{T}_d(k_e, k_v)$  if  $\Lambda = (6k_e\lambda_e, 4k_v\lambda_v)$  and so  $N = 4, 6, 8, 10$  or  $14$ ;
3.  $\mathbb{O}_h(k_e, k_f, k_v)$  if  $\Lambda = (12k_e\lambda_e, 8k_f\lambda_f, 6k_v\lambda_v)$  and so  $N = 6, 8, 12, 14, 18, 20$  or  $26$ ;
4.  $\mathbb{I}_h(k_e, k_f, k_v)$  if  $\Lambda = (30k_e\lambda_e, 20k_f\lambda_f, 12k_v\lambda_v)$  and so  $N = 12, 20, 30, 32, 42, 50$  or  $62$ .

PROOF The strata listed are precisely those which consist of isolated  $\mathbf{SO}(3)$  orbits.  $\square$

**Corollary 2.4** *For  $N$  identical vortices with  $N = 3, \dots, 12$  there exist equilibrium points in each of the strata listed in the second and third columns of Table 4.*

## 2.2 Relative equilibria

The following result is completely analogous to the existence theorem for equilibrium points in the previous section.

**Theorem 2.5** *Let  $X \subset \mathcal{P}$  be an orbit type stratum of the action of either  $G$  or  $\widehat{G}$  on  $\mathcal{P}$ , and write  $X_\mu = X \cap \Phi^{-1}(\mu)$ .*

1. *If a connected component of  $X_\mu$  is contained in a single  $\mathbf{SO}(3)$  orbit then it is a relative equilibrium configuration of every invariant Hamiltonian on  $\mathcal{P}$ .*
2. *If  $H(\mathbf{x}) \rightarrow \infty$  as  $\mathbf{x} \rightarrow \Delta$  then there exists at least one relative equilibrium point on the closure of each connected component of  $X_\mu$  in  $\Phi^{-1}(\mu)$ . In particular, if  $X_\mu$  is minimal in  $\Phi^{-1}(\mu)$  then there exists at least one relative equilibrium in  $X_\mu$ .*

PROOF The relative equilibria in  $\Phi^{-1}(\mu)$  are precisely the critical points of the restriction of  $H$  to  $\Phi^{-1}(\mu)$ . The proof of this result therefore follows that of Theorem 2.2, but with  $H$  replaced by  $H|_{\Phi^{-1}(\mu)}$ .  $\square$

In particular it follows that if all the vorticities have the same sign then there will be at least one relative equilibrium with centre of vorticity  $\mu$  in the closure of each stratum that intersects  $\Phi^{-1}(\mu)$ , and at least one in any minimal stratum. Equivalently we can say that there must be at least one  $\widehat{G}$  orbit of relative equilibria in the closure of the corresponding orbit type stratum in the quotient space. It should be pointed out that some of these relative equilibria may in fact be equilibria.

Recall from §1.4 that a zero-momentum stratum is one entirely contained in the level-set  $\Phi^{-1}(0)$ .

**Proposition 2.6** *Any relative equilibrium in a zero-momentum stratum is an equilibrium point.*

PROOF Let  $X$  be a zero-momentum stratum and  $p \in X$  be a relative equilibrium with  $\xi \in \mathfrak{so}(3)$  the corresponding angular velocity at  $p$ . Let  $\Gamma \subset \widehat{G}$  be the symmetry group of the point  $p$ . Then by Proposition 2.1,  $\xi \in \text{Fix}(\Gamma, \mathfrak{so}(3))$ . Now  $\text{Fix}(\Gamma, \mathfrak{so}(3)^*) = 0$  because  $X$  is a zero-momentum stratum, and so  $\text{Fix}(\Gamma, \mathfrak{so}(3)) = 0$  and  $\xi = 0$ .  $\square$

Thus it is only possible to obtain relative equilibria which are not equilibrium points in the  $\mathbf{C}_{nv}$ ,  $\mathbf{C}_h$   $\mathbf{C}_n$  and  $\mathbf{1}$  strata. We discuss the consequences of Theorem 2.5 for each of these in turn.

### $\mathbf{C}_n$ relative equilibria

Recall from Proposition 1.6 that  $\mathbf{C}_n(k_R, k_p)$  corresponds to an isotropy subgroup of  $\widehat{G}$  if and only if  $k_R > 1$  and  $k_p = 0, 1$  or  $2$ . The quotient of the intersection of the stratum  $\mathbf{C}_n(k_R, k_p)$  with any level set  $\Phi^{-1}(\mu)$  is connected. If all the vorticities have the same sign then the theorem states that there will be at least one  $\widehat{G}$  orbit of relative equilibria in the closure of  $\mathbf{C}_n(k_R, k_p)$  for each  $\mu$  for which the intersection is nonempty. However the frontier of  $\mathbf{C}_n(k_R, k_p)$  always contains  $\mathbf{C}_{nv}$  strata in which there must also be relative equilibria, so the theorem does not prove the existence of relative equilibria with precisely  $\mathbf{C}_n$  symmetry. In §3.1 we prove that for the standard point vortex system the stratum  $\mathbf{C}_n(k_R, k_p)$  with  $k_R = 2$  never contains relative equilibria. However it seems likely that relative equilibria with a larger number of rings and symmetry group precisely  $\mathbf{C}_n$  do exist.

### $\mathbf{C}_{nv}$ relative equilibria

The  $\mathbf{C}_{nv}(k_R, k_{R'}, k_p)$  configurations consist of  $k_R$  aligned ‘latitudinal’  $n$ -rings, another set of  $k_{R'}$  aligned rings which are offset by  $\pi/n$  with respect to the first set, and  $k_p = 0, 1$  or  $2$  polar vorticities. If the vorticities of the rings are all distinct then the quotients of the intersections of these strata with  $\Phi^{-1}(\mu)$  are either empty or have  $k_R!k_{R'}$  connected components, one for each ordering of the  $k_R$   $n$ -rings and the  $k_{R'}$  dual  $n$ -rings. If some of the vorticities are equal then the extra permutational symmetries identify some of these strata, and if they are all equal then there is a unique component. The theorem states that if all the vorticities have the same sign then there will be a  $\widehat{G}$  orbit of relative equilibria in the closure of each connected component. In some cases these relative equilibria will have higher symmetries, but in many cases the  $\mathbf{C}_{nv}(k_R, k_{R'}, k_p)$  are minimal and so the relative equilibria will have precisely  $\mathbf{C}_{nv}$  symmetry. More detailed discussions of the relative equilibria of types  $\mathbf{C}_{nv}(R, k_p)$ ,  $\mathbf{C}_{nv}(2R, k_p)$  and  $\mathbf{C}_{nv}(R, R', k_p)$  for the point-vortex Hamiltonian (1.2) are given in §3.1. There it is shown that it is possible for these relative equilibria to have  $\mu = 0$  and  $\xi \neq 0$ , and vice-versa. This does not occur for simple mechanical systems with Hamiltonians which are the sums of potential and kinetic energy terms.

### $\mathbf{C}_h$ relative equilibria

If all the ‘equatorial’ vorticities are distinct, the  $\mathbf{C}_h(k_E, k_R)$  strata in the orbit space  $\mathcal{P}/\widehat{G}$  have  $\frac{1}{2}(k_E - 1)!$  connected components, one for each cyclic ordering of the ‘equatorial’ points. For the point vortex symplectic form (1.1) its intersections with the level sets of  $\Phi$  are either empty or again have  $\frac{1}{2}(k_E - 1)!$  connected components and there will be a  $\widehat{G}$  orbit of relative equilibria in the closure of each of these if all the vorticities have the same sign. In §3.2 a convexity argument is used to show that for the specific point-vortex Hamiltonian (1.2) there is a *unique* equilibrium point ( $\xi = 0$ , with  $\mu$  unspecified) in each of these strata.

### 2.3 Bifurcation from zero centre of vorticity

In this section we describe some results that can be obtained by adapting the ideas of Montaldi and Roberts [31, 32] to systems of point vortices. These results give a method for finding some branches of relative equilibria which do not have maximal symmetry. The branches can then be followed using numerical methods.

Since the action of  $\mathbf{SO}(3)$  is free we can apply the methods developed in [31, 32] to analyse the bifurcations of relative equilibria. We briefly recall the principal ideas. Let  $\mathbf{x} \in \mathcal{P}$ , and let  $S$  be a slice to the  $\mathbf{SO}(3)$ -action at  $\mathbf{x}$ , so that:

$$T_{\mathbf{x}}S \oplus \mathfrak{g} \cdot \mathbf{x} = T_{\mathbf{x}}\mathcal{P}$$

where  $\mathfrak{g} = \mathfrak{so}(3)$ . If the isotropy subgroup  $\widehat{G}_{\mathbf{x}}$  is non-trivial, we choose  $S$  to be  $\widehat{G}_{\mathbf{x}}$ -invariant. The Guillemin-Sternberg-Marle normal form [11, 26] for symplectic group actions shows that if  $\Phi(\mathbf{x}) = 0$ , then in a neighbourhood of the point  $\mathbf{x}$

$$S \simeq \mathcal{P}_0 \times \mathfrak{g}^*, \quad (2.2)$$

where  $\mathcal{P}_0$  is (a neighbourhood of  $\mathbf{x}$  in) the zero reduced space. For  $\mu$  close to zero in  $\mathfrak{so}(3)^*$ , the  $\mu$ -reduced space is then

$$\mathcal{P}_{\mu} \simeq \mathcal{P}_0 \times \mathcal{O}_{\mu},$$

where  $\mathcal{O}_{\mu}$  is the orbit of the coadjoint action of  $\mathbf{SO}(3)$  on  $\mathfrak{so}(3)^*$  containing  $\mu$ . For  $\mathfrak{g} = \mathfrak{so}(3)$  this is just the sphere centre 0 radius  $|\mu|$ .

Since  $H$  is  $\widehat{G}$ -invariant, it restricts to  $S$  in a way that is independent of the choice of  $S$ . Identifying  $S$  with the product in (2.2), we write  $H : \mathcal{P}_0 \times \mathfrak{g}^* \rightarrow \mathbf{R}$ . A point  $(\mathbf{y}, \nu) \in \mathcal{P}_0 \times \mathfrak{g}^*$  is a relative equilibrium if the restriction  $H_{\nu}$  of  $H$  to  $\mathcal{P}_{\nu}$  has a critical point at that point. Moreover, the relative equilibrium is *non-degenerate* if that critical point is non-degenerate.

Suppose  $x \in \mathcal{P}_0$  is a non-degenerate relative equilibrium; that is,  $H_0$  has a non-degenerate critical point at  $x$ . Then the differential  $dH_{(x,0)}$  is a linear map from  $\mathfrak{g}^*$  to  $\mathbf{R}$  and so naturally an element of  $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ . It is shown in [31] that under this identification  $dH_{(x,0)} = \xi$ , the angular velocity of the relative equilibrium. In particular, if  $\xi \neq 0$  then on each nearby reduced space there are precisely 2 relative equilibria (which by a theorem of G. Patrick [35] form a smooth curve in  $\mathcal{P}_0 \times \mathfrak{g}^*$ ). On the other hand, if  $\xi = 0$ , then there will be more relative equilibria on each nearby reduced space.

The general procedure for analyzing this bifurcation is to use the splitting lemma (or alternatively Lyapounov-Schmidt reduction), which says that there is a  $G_{\mathbf{x}}$ -equivariant diffeomorphism  $\Psi : \mathcal{P}_0 \times \mathfrak{g}^* \rightarrow \mathcal{P}_0 \times \mathfrak{g}^*$  (in a neighbourhood of the point  $(x, 0)$ ) of the form

$$\Psi(y, \mu) = (\psi(y, \mu), \mu),$$

for which

$$H \circ \Psi(y, \mu) = Q(y) + h(\mu), \quad (2.3)$$

where  $Q$  is a non-degenerate quadratic form, and  $h$  is a smooth  $G_{\mathbf{x}}$ -invariant function on  $\mathfrak{g}^*$ . It follows that  $(y_{\mu}, \mu) \in \mathcal{P}_0 \times \mathfrak{g}^*$  is a relative equilibrium for  $H$  if, and only if,  $\mu$  is a critical point of the restriction  $h_{\mu}$  of  $h$  to the sphere  $\mathcal{O}_{\mu}$ , and  $y_{\mu} = \psi(0, \mu)$ .

Thus the relative equilibria near  $\mathbf{x}$  with momentum value  $\mu$  are in 1-1 correspondence with the critical points of the restriction  $h_{\mu}$  of a  $G_{\mathbf{x}}$ -invariant function  $h$  on  $\mathfrak{g}^*$ . Moreover  $dh_0 = dH_{(0,0)} = \xi$ , the angular velocity of the relative equilibrium  $\mathbf{x}$ . See [32, Theorem 2.7] for a more detailed statement.

**Theorem 2.7** *Let  $\Sigma$  be a subgroup of  $\widehat{G}$ , and  $\mathbf{x}$  a non-degenerate relative equilibrium in  $\text{Fix}(\Sigma, \mathcal{P})$ , with  $\Phi(\mathbf{x}) = 0$ , and let  $\Gamma = \pi(\Sigma) < \mathbf{O}(3)$ . Consider the usual action of  $\Gamma$  on  $S^2 \subset \mathbf{R}^3$ . Let  $\Gamma' < \Gamma$  be such that  $\text{Fix}(\Gamma', S^2) \neq \emptyset$  and  $\Sigma' = \pi^{-1}(\Gamma') \cap \Sigma$ . Then there is a relative equilibrium near  $\mathbf{x}$  with symmetry group containing  $\Sigma'$ .*

*Remarks*

- (1) If  $\text{Fix}(\Gamma, \mathfrak{g}) \neq \{0\}$  then  $\text{Fix}(\Gamma, S^2) \neq \emptyset$ . If furthermore  $\xi \neq 0$  then the only critical points will be those with symmetry  $\Gamma$ . The interesting case is therefore when  $\xi = 0$ .
- (2) A particularly useful case of the theorem is when  $\text{Fix}(\Gamma, \mathfrak{g}) = 0$ , and so  $\xi = 0$ . Then, if  $\Gamma' < \Gamma$  is a maximal isotropy subgroup of the  $\Gamma$  action on  $S^2$ , there is a relative equilibrium with symmetry precisely  $\Sigma' = \pi^{-1}(\Gamma') \cap \Sigma$ .
- (3) A more precise statement of the relation between  $\Sigma$  and  $\Sigma'$  can be formulated in terms of orbit types as follows. Suppose  $\Gamma$  is a subgroup of  $\mathbf{O}(3)$  and a system of point vortices on  $S^2$  has a non-degenerate equilibrium of type  $\Gamma(k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d)$  with  $\mu = 0$ . If  $\Gamma' < \Gamma$  then the corresponding isotropy subgroup  $\Gamma'(k'_1\mathcal{O}'_1, \dots, k'_e\mathcal{O}'_e)$  is the orbit type decomposition obtained by restricting the action of  $\Gamma$  on  $k_1\mathcal{O}_1 \cup \dots \cup k_d\mathcal{O}_d$  to  $\Gamma'$ .  $\diamond$

**PROOF** The theorem follows from the discussion above by the principle of symmetric criticality. The relative equilibria correspond to the critical points of the restriction  $h_\mu$  of a smooth  $\Gamma$ -invariant function  $h$ , and if  $\Gamma' \subset \Gamma$  then  $h_\mu$  restricted to  $\text{Fix}(\Gamma')$  must have a critical point.

The relationship between  $\Sigma$  and  $\Gamma$  arises because the action of  $\widehat{G}$  on  $\mathfrak{g}^*$  factors through that of  $\mathbf{O}(3)$ . Moreover, since the permutation group  $S(\Lambda)$  acts freely on  $\mathcal{P}$ , the projection  $\pi : \widehat{G} \rightarrow \mathbf{O}(3)$  gives an isomorphism when restricted to any isotropy subgroup  $\Sigma$ .  $\square$

For each stratum  $\Gamma(k_1\mathcal{O}_1, \dots, k_d\mathcal{O}_d)$  a more detailed analysis can yield further information on the relative equilibria bifurcating from an equilibrium point. As an example we show that an *equilibrium* point in a  $\mathbf{C}_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$  stratum has more than just the  $\mathbf{C}_{nv}$  relative equilibria predicted by Theorem 2.7 bifurcating from it.

Let  $\varphi : \mathcal{P} \rightarrow \mathbf{R}$  be the function  $\varphi(\mathbf{x}) = \|\Phi(\mathbf{x})\|^2$ , where  $\|\cdot\|$  is an  $\mathbf{SO}(3)$ -invariant norm on  $\mathfrak{so}(3)^*$  (unique up to scalar multiple). Being  $\widehat{G}$ -invariant, the restriction of  $\varphi$  to a slice is independent of the choice of slice. Under the splitting described above,  $\varphi : S \times \mathfrak{so}(3)^* \rightarrow \mathbf{R}$  takes the form  $\varphi(y, \mu) = |\mu|^2$ , so that  $d^2\varphi$  is of rank 3.

**Proposition 2.8** *Suppose a system of point vortices on  $S^2$  has a non-degenerate equilibrium  $\mathbf{x}$  (so  $\xi = 0$ ) of type  $\mathbf{C}_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$  with  $\mu = 0$ , and suppose furthermore that the (cubic) polynomial in  $\alpha$*

$$\det [d_S^2(H - \alpha\varphi)(\mathbf{x})] \quad (2.4)$$

*has precisely 2 distinct roots, where  $d_S^2$  denotes the second differential restricted to any slice to the group orbit. Then for each of the types*

1.  $\mathbf{C}_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$  ( $m = 2$ );
2. ( $n$  odd)  $\mathbf{C}_h(k'_R, k'_E)$  with  $k'_R = \frac{(n-1)}{2}(k_R + k_{R'}) + nk_{R_s}$ ,  $k'_E = k_R + k_{R'} + k_p$  ( $m = 2$ );
3. ( $n$  even)  $\mathbf{C}_h(k'_R, k'_E)$  with  $k'_R = \frac{(n-2)}{2}k_R + \frac{n}{2}k_{R'} + nk_{R_s}$ ,  $k'_E = 2k_R + k_p$  ( $m = 1$ );
4. ( $n$  even)  $\mathbf{C}_h(k'_R, k'_E)$  with  $k'_R = \frac{n}{2}k_R + \frac{(n-2)}{2}k_{R'} + nk_{R_s}$ ,  $k'_E = 2k_{R'} + k_p$  ( $m = 1$ );

*there exists  $\varepsilon > 0$  and precisely  $m$  inequivalent relative equilibria of that type in  $\Phi^{-1}(\mu)$  for each  $\mu$  satisfying  $0 < |\mu| < \varepsilon$ .*

**Remark 2.9** That (2.4) is a cubic polynomial in  $\alpha$  follows from the fact that  $d^2\varphi(\mathbf{x})$  is of rank 3. Without the  $\mathbf{C}_{nv}$  or similar symmetry this polynomial will generically have 3 distinct roots, in which case there are

6 families of bifurcating relative equilibria. For example, for the free rigid body the three roots will be the inverses of the principal moments of inertia (up to a factor depending on the choice of norm on  $\mathfrak{so}(3)^*$ ). This occurs not only for equilibria with trivial symmetry, but also for those with symmetry contained in  $\mathbf{D}_{2h}$ .

On the other hand, if the equilibrium has cubic symmetry ( $\mathbb{T}$ ,  $\mathbb{O}$  or  $\mathbb{I}$ ) then (2.4) will only have a single root. For example, this occurs for the  $\mathbf{C}_{3v}(R, p)$  equilibrium with 4 identical vortices, which in fact has symmetry  $\mathbb{T}_d(v)$ . In this case one has to look at the higher order derivatives of  $H$ .  $\diamond$

**PROOF** Let  $S$  be any slice to the group orbit at  $\mathbf{x}$ , and let  $Y = \Phi^{-1}(0) \cap S$ . Then  $Y \simeq \mathcal{P}_0$  (the zero reduced space) and since the equilibrium is non-degenerate, the restriction to  $Y$  of the Hessian matrix  $d^2H$  of  $H$  is non-degenerate. We can therefore use the quadratic form  $d^2H$  to split  $S$  as a product  $S = Y \times \mathfrak{so}(3)^*$ , in such a manner that  $d_S^2H$  is block-diagonalized. Then by [32, Theorem 2.7], the diffeomorphism  $\Psi$  arising from the splitting lemma can be chosen to have linear part at  $\mathbf{x}$  equal to the identity. Differentiating (2.3) twice at  $\mathbf{x}$  shows that

$$d_S^2H(\mathbf{x}) = \begin{bmatrix} 2Q & 0 \\ 0 & d^2h(0) \end{bmatrix},$$

so that

$$\det [d_S^2(H - \alpha\varphi)(\mathbf{x})] = 2^{\dim Y} \det Q \det [d^2(h - \alpha\bar{\varphi})(0)],$$

where  $\bar{\varphi}$  is the norm function  $\mu \rightarrow \|\mu\|^2$ . By hypothesis,  $Q$  is non-degenerate, so the genericity hypothesis of the theorem is equivalent to

$$\det [d^2(h - \alpha\bar{\varphi})(0)] \tag{2.5}$$

having precisely two roots.

As explained above, the relative equilibria near  $\mathbf{x}$  correspond to critical points of  $h$  restricted to the coadjoint orbits (spheres)  $\mathcal{O}_\mu$ , which are the level sets of  $\bar{\varphi}$ . The  $\mathbf{C}_{nv}$  symmetry of the equilibrium means that  $h$  and  $\bar{\varphi}$  are  $\mathbf{C}_{nv}$ -invariant functions. Recall that  $\mathbf{C}_{nv}$  symmetry is dihedral  $D_n$  symmetry in the  $x - y$  plane in  $\mathfrak{so}(3)^* \simeq \mathbf{R}^3$ . On each sphere  $\mathcal{O}_\mu$ , the fixed point set  $\text{Fix}(\mathbf{C}_{nv}; \mathcal{O}_\mu)$  consists of two points – the poles at  $x = y = 0$ . These are therefore critical points of  $h_\mu := H|_{\mathcal{O}_\mu}$  and, being of maximal symmetry type, were predicted by Theorem 2.7 above. Choosing coordinates so that  $\mathbf{C}_{nv}$  acts as described and  $\bar{\varphi}(x, y, z) = x^2 + y^2 + z^2$ , we can write the Taylor series at 0 of  $h$  to order 2 as

$$h(x, y, z) = a(x^2 + y^2) + bz^2 + f(x, y, z), \tag{2.6}$$

where  $f$  is of order 3. The roots of (2.5) are then  $\alpha = a$  (double) and  $\alpha = b$  (simple), so the genericity hypothesis is simply that  $a \neq b$ .

For each subgroup  $\mathbf{C}_h$  of  $\mathbf{C}_{nv}$ , the fixed point space  $\text{Fix}(\mathbf{C}_h; \mathcal{O}_\mu)$  is a circle containing the two poles. If  $n$  is even there are 2 distinct inequivalent such circles, while if  $n$  is odd there is only one (‘inequivalent’ under symmetry operations), but the argument in each case is the same.

Restricting  $h$  to  $\text{Fix}(\mathbf{C}_h; \mathcal{O}_\mu)$  gives a function on the circle, which has critical points at each of the poles, and we wish to show that if  $a \neq b$  it has two further critical points. We do this by a blowing-up argument very similar to the one used in [31]. Rotate the  $x - y$  plane so that the circle is given by  $y = 0$ . Then on  $\text{Fix}(\mathbf{C}_h; \mathfrak{so}(3)^*)$  we have

$$h(x, z) = ax^2 + bz^2 + f(x, 0, z),$$

with  $a \neq b$ . In polar coordinates, we can write  $h(r, \theta) = r^2(a \cos^2 \theta + b \sin^2 \theta) + r^3 \tilde{f}(r, \theta)$ , or

$$\frac{1}{r^2}h(r, \theta) = a \cos^2 \theta + b \sin^2 \theta + r \tilde{f}(r, \theta),$$

where  $\tilde{f}$  is a smooth function of  $r, \theta$ . Note that for  $r \neq 0$  the critical points of the restrictions to  $\mathcal{O}_\mu$  of  $h$  and of  $h/r^2$  coincide. Differentiating  $h$  with respect to  $\theta$  gives

$$\frac{1}{r^2} \frac{\partial h}{\partial \theta} = (b - a) \sin(2\theta) + r \tilde{f}'(r, \theta),$$

where  $\tilde{f}' = \partial \tilde{f} / \partial \theta$ . At  $r = 0$  this has 4 non-degenerate zeros, so by the implicit function theorem and the compactness of the circle, it also has 4 non-degenerate zeros for sufficiently small values of  $r$ .  $\square$

Table 5 lists the  $\Gamma'(k'_1 \mathcal{O}'_1, \dots, k'_e \mathcal{O}'_e)$  for which Theorem 2.7 and Proposition 2.8 give bifurcating relative equilibria for each  $\Gamma(k_1 \mathcal{O}_1, \dots, k_d \mathcal{O}_d)$  strata with  $\Gamma = \mathbf{C}_{nv}, \mathbf{D}_{nh}, \mathbb{T}_d, \mathbb{O}_h$  and  $\mathbb{I}_h$ .

Stabilities of these bifurcating relative equilibria can often be determined with the help of [32, Theorem 2.8]. In particular, if the equilibrium is *extremal* (for example if  $Q$  is positive definite), then the bifurcating relative equilibria for which the constructed function  $h$  is minimal, when restricted to the sphere  $\mathcal{O}_\mu$ , is then also extremal, and so Lyapounov stable. Note that in terms of the Taylor series (2.6), if  $a < b$  then the relative equilibria with symmetry  $\mathbf{C}_{nv}$  are of lower energy, and so (both) stable, while if  $b > a$  then at least one of the bifurcating relative equilibria of type  $\mathbf{C}_h$  is Lyapounov stable. This argument will be pursued in the forthcoming paper [24].

### 3 Rings and Poles

In this section we describe some results that depend on the explicit form of the point vortex Hamiltonians (1.2), rather than just on its symmetry properties. In particular we derive algebraic equations for relative equilibria (Proposition 3.3) and then use these to discuss relative equilibria which consist of either one (§3.1.1) or two (§3.1.2) rings of identical vortices, possibly with extra ‘polar’ vortices. In the two ring case we show that the rings must be either ‘aligned’ or ‘staggered’ (Proposition 3.6). In §3.2 we give an existence and uniqueness result for *equilibria* where all the vortices lie in a great circle.

#### 3.1 Relative equilibria with $\mathbf{C}_n$ and $\mathbf{C}_{nv}$ symmetry

By Proposition 2.6 any relative equilibrium which is not an equilibrium point must lie in a  $\mathbf{C}_n, \mathbf{C}_{nv}, \mathbf{C}_h$  or  $\mathbf{1}$  stratum ( $n \geq 1$ ). In this section we will obtain some results on the relative equilibria in the  $\mathbf{C}_n$  and  $\mathbf{C}_{nv}$  strata.

A relative equilibrium lies in the closure of a  $\mathbf{C}_n$  stratum if and only if the point vortices form a number,  $k$  say, of ‘latitudinal’ regular  $n$ -rings, together with 0, 1 or 2 polar vortices. We will assume that the rings lie in planes perpendicular to the  $z$ -axis and so the poles lie on the  $z$ -axis. The angular velocity can therefore be written as  $(0, 0, \xi)$  and the centre of vorticity as  $(0, 0, \mu)$ . Let  $\lambda_j$  denote vorticity of the vortices in the  $i$ -th ring and  $\lambda_n$  and  $\lambda_s$  the polar vorticities. We will assume that the  $\lambda_i$  are all non-zero, but allow  $\lambda_n$  and  $\lambda_s$  to be zero. Denote the co-latitude of the  $i$ -th ring by  $\theta_i$ . Fix a particular vortex in each ring and denote its longitude by  $\phi_i$ . Let  $\phi_{ij} = \phi_i - \phi_j$  (which is well-defined modulo  $2\pi/n$ ). Note that

$$\mu = \lambda_n - \lambda_s + \sum_{j=1}^k \lambda_j \cos \theta_j. \quad (3.1)$$

**Lemma 3.1** *The Hamiltonian (1.2) for these configurations is given by:*

$$H = \lambda_n \lambda_s \log 2 - \sum_{1 \leq i \leq k} \left( \lambda_i^2 \overline{H}_i + \lambda_i \lambda_n \overline{H}_i^n + \lambda_i \lambda_s \overline{H}_i^s \right) - \sum_{1 \leq i < j \leq k} \lambda_i \lambda_j \overline{H}_{ij} \quad (3.2)$$

Equilibrium	Bifurcating relative equilibria	No.	Orbit relations
$C_{nv}$	$C_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$	1, 1	
	$C_h(k'_{R'}, k'_E)$ $n$ odd	$n, n$	$k'_{R'} = \frac{(n-1)}{2}(k_R + k_{R'}) + nk_{R_s}$ $k'_E = k_R + k_{R'} + k_p$
	$C_h(k'_{R'}, k'_E)$ $n$ even	$n$	$k'_{R'} = \frac{(n-2)}{2}k_R + \frac{n}{2}k_{R'} + nk_{R_s}$ $k'_E = 2k_R + k_p$
	$C_h(k'_{R'}, k'_E)$ $n$ even	$n$	$k'_{R'} = \frac{n}{2}k_R + \frac{(n-2)}{2}k_{R'} + nk_{R_s}$ $k'_E = 2k_{R'} + k_p$
$D_{nh}$	$C_{nv}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	2	$k'_{R_s} = 2k_{R_s} + k_{R_s^e}, k'_p = 2k_p$ $k'_{R'} = 2k_R + k_r, k'_{R'} = 2k_{R'} + k_{r'}$
	$C_{2v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$ $n$ odd	$n, n$	$k'_{R_s} = nk_{R_s} + \frac{(n-1)}{2}(k_R + k_{R'})$ $k'_{R'} = nk_{R_s^e} + \frac{(n-1)}{2}(k_r + k_{r'})$ $k'_p = k_R + k_{R'} + k_p, k'_p = k_r + k_{r'}$
	$C_{2v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$ $n$ even	$n$	$k'_{R_s} = nk_{R_s} + \frac{(n-2)}{2}k_R + \frac{n}{2}k_{R'}$ $k'_{R'} = nk_{R_s^e} + \frac{(n-2)}{2}k_r + \frac{n}{2}k_{r'}$ $k'_p = 2k_R + k_p, k'_p = 2k_r$
	$C_{2v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$ $n$ even	$n$	$k'_{R_s} = nk_{R_s} + \frac{n}{2}k_R + \frac{(n-2)}{2}k_{R'}$ $k'_{R'} = nk_{R_s^e} + \frac{n}{2}k_r + \frac{(n-2)}{2}k_{r'}$ $k'_p = 2k_{R'} + k_p, k'_p = 2k_{r'}$
$T_d$	$C_{3v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	4, 4	$k'_{R_s} = 4k_R + k_E, k'_p = k_v$ $k'_{R'} = 2k_E + k_e + k_v, k'_{R'} = k_e$
	$C_{2v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	6	$k'_{R_s} = k_e + 2k_E + 6k_R, k'_p = 2k_e$ $k'_{R'} = k'_{R'} = k_E + k_v$
$O_h$	$C_{4v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	6	$k'_{R_s} = 6k_R + k_E + 2k_{E'}, k'_p = 2k_v$ $k'_{R'} = 4k_E + 2k_e + k_v$ $k'_{R'} = 2k_{E'} + k_e + 2k_f$
	$C_{3v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	8	$k'_{R_s} = 8k_R + 4k_E + 2k_{E'} + k_e, k'_p = 2k_f$ $k'_{R'} = k'_{R'} = 2k_{E'} + k_e + k_f + k_v$
	$C_{2v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	12	$k'_{R_s} = 12k_R + 4k_E + 5k_{E'}, k'_p = 2k_e$ $k'_{R'} = 4k_E + k_e + 2k_v,$ $k'_{R'} = 2k_{E'} + 2k_f + k_v$
$I_h$	$C_{5v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	12	$k'_{R_s} = 12k_R + 4k_E + 4k_{E'} + k_e, k'_p = 2k_v$ $k'_{R'} = k'_{R'} = 2k_E + 2k_{E'} + 2k_e + 2k_f + k_v$
	$C_{3v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	20	$k'_{R_s} = 20k_R + 8k_E + 8k_{E'} + 4k_e + 2k_f$ $k'_p = 2k_f$
	$C_{2v}(k'_{R_s}, k'_{R'}, k'_{R'}, k'_p)$	30	$k'_{R_s} = k'_{R'} = 2k_E + 2k_{E'} + k_e + k_f + 2k_v$ $k'_{R_s} = 30k_R + 13k_E +$ $\quad + 13k_{E'} + 6k_e + 3k_f + k_v$ $k'_{R'} = k'_{R'} = 2k_E + k_e + 2k_f + 2k_v, k'_p = 2k_e$

Table 5: Types of relative equilibria bifurcating from equilibria with  $\mu = 0$ , as given by Theorem 2.7 and Proposition 2.8. In the first column, the orbit type is understood, for example  $C_{nv} = C_{nv}(k_{R_s}, k_R, k_{R'}, k_p)$ .

where

$$\begin{aligned}\overline{H}_i &= n \log n - \frac{1}{2}n(n-1) \log 2 + n(n-1) \log \sin \theta_i \\ \overline{H}_i^n &= n \log(1 - \cos \theta_i) \\ \overline{H}_i^s &= n \log(1 + \cos \theta_i) \\ \overline{H}_{ij} &= n \log(C(\theta_i, \theta_j) + D(\theta_i, \theta_j) \cos(n\phi_{ij}))\end{aligned}$$

and

$$D(\theta_i, \theta_j) = \frac{1}{2^{n-3}} (\sin \theta_i \sin \theta_j)^n \quad (3.3)$$

$$\begin{aligned}C(\theta_i, \theta_j) &= \frac{1}{2} \prod_{1 \leq \alpha \leq n} \left( 1 - \cos \theta_i \cos \theta_j - \sin \theta_i \sin \theta_j \cos \frac{2\alpha\pi}{n} \right) \\ &\quad + \frac{1}{2} \prod_{1 \leq \alpha \leq n} \left( 1 - \cos \theta_i \cos \theta_j - \sin \theta_i \sin \theta_j \cos \frac{(2\alpha+1)\pi}{n} \right).\end{aligned} \quad (3.4)$$

**PROOF** This consists of calculations, all of which are straightforward, except possibly the expression for  $\overline{H}_{ij}$ . For this we have  $\overline{H}_{ij} = \log P_{ij}$  where

$$P_{ij} = \prod_{1 \leq \alpha, \beta \leq n} \left( 1 - \sin \theta_i \sin \theta_j \cos \left( \phi_{ij} + 2(\alpha - \beta) \frac{\pi}{n} \right) - \cos \theta_i \cos \theta_j \right).$$

The right hand side of this equation for  $P_{ij}$  is a trigonometric polynomial of degree  $n$  in  $\phi_{ij}$ , and so can be written as a linear combination of  $\sin \ell \phi_{ij}$  and  $\cos \ell \phi_{ij}$  for  $\ell = 1 \dots n$ , the coefficients being trigonometric polynomials in  $\theta_i$  and  $\theta_j$ . However it also follows from the formula that  $P_{ij}$  must be invariant under the translation  $\phi_{ij} \mapsto \phi_{ij} + 2\pi/n$  and the reflection  $\phi_{ij} \mapsto -\phi_{ij}$ . Thus  $P_{ij} = C + D \cos n\phi_{ij}$ . Substituting  $\phi_{ij} = 0$  and  $\pi/n$ , respectively, in this expression gives

$$C + D = \prod_{1 \leq \alpha \leq n} \left( 1 - \cos \theta_i \cos \theta_j - \sin \theta_i \sin \theta_j \cos \frac{2\alpha\pi}{n} \right) \quad (3.5)$$

$$C - D = \prod_{1 \leq \alpha \leq n} \left( 1 - \cos \theta_i \cos \theta_j - \sin \theta_i \sin \theta_j \cos \frac{(2\alpha+1)\pi}{n} \right) \quad (3.6)$$

The expression for  $C$  follows immediately, while that for  $D$  follows from Lemma 3.2 below, after taking  $(\sin \theta_i \sin \theta_j)^n$  as a factor and substituting  $x = (1 - \cos \theta_i \cos \theta_j) / (\sin \theta_i \sin \theta_j)$ .  $\square$

**Lemma 3.2** *Let*

$$\begin{aligned}D_1(x) &= \prod_{\alpha=1}^n (x - \cos(2\alpha\pi/n)) \\ D_2(x) &= \prod_{\alpha=1}^n (x - \cos((2\alpha+1)\pi/n)).\end{aligned}$$

Then

$$D_1(x) - D_2(x) = -\frac{1}{2^{n-2}}.$$

PROOF Let  $x = \cos \theta$ . If the equality can be established for  $x = \cos \theta$  (i.e.  $x \in [-1, 1]$ ) the result follows, for the expressions  $D_j(x)$  are polynomials in  $x$ . We claim that

$$D_1(\cos \theta) = \frac{\cos(n\theta) - 1}{2^{n-1}}, \quad D_2(\cos \theta) = \frac{\cos(n\theta) + 1}{2^{n-1}},$$

from which the result then follows.

To prove the claim, first note that

$$\prod_{\alpha=1}^n \left( \cos \theta - \cos \left( \frac{2\alpha\pi}{n} \right) \right) = 2^n \prod_{\alpha=1}^n \left[ \sin \left( \frac{\alpha\pi}{n} + \frac{\theta}{2} \right) \sin \left( \frac{\alpha\pi}{n} - \frac{\theta}{2} \right) \right],$$

and this expression is clearly  $(2\pi/n)$ -periodic in  $\theta$ . Consequently, writing  $D_1(\cos \theta)$  as a Fourier polynomial gives

$$D_1(\cos \theta) = a_0 + a \cos(n\theta) + b \sin(n\theta).$$

The parity of the expression shows that  $b = 0$  and the fact that it vanishes for  $\theta = 2\alpha\pi/n$  shows that  $a_0 + a = 0$ . The expansion of the product  $D_1(\cos \theta)$  is a polynomial in  $\cos \theta$  of degree  $n$ , whose coefficient of  $\cos^n \theta$  is 1. The expansion of  $\cos(n\theta)$  in terms of powers of  $\cos(\theta)$  has leading term  $2^{n-1} \cos^n \theta$ , so that a comparison of the coefficients shows that  $a = 2^{1-n} = -a_0$ . A similar argument works for  $D_2(\cos \theta)$ .  $\square$

**Proposition 3.3** *A system of  $k$  parallel regular  $n$ -rings is a relative equilibrium with angular velocity  $\xi$  if and only if the co-latitudes  $\theta_i$  and longitudes  $\phi_i$  satisfy the equations*

$$\sum_{j=1}^k A_{ij}(\theta, \phi) \lambda_j = 0 \quad (3.7)$$

$$\sum_{j=1}^k B_{ij}(\theta, \phi) \lambda_j = \xi \sin \theta_i - \lambda_n \frac{\sin \theta_i}{1 - \cos \theta_i} + \lambda_s \frac{\sin \theta_i}{1 + \cos \theta_i} \quad (3.8)$$

for  $i = 1 \dots k$ , where

$$A_{ii}(\theta, \phi) = 0 \quad (3.9)$$

$$A_{ij}(\theta, \phi) = -\frac{1}{n^2} \frac{\partial \bar{H}_{ij}}{\partial \phi_i} = \frac{D(\theta_i, \theta_j) \sin n\phi_{ij}}{C(\theta_i, \theta_j) + D(\theta_i, \theta_j) \cos n\phi_{ij}} \quad i \neq j \quad (3.10)$$

$$B_{ii}(\theta, \phi) = -\frac{1}{n} \frac{\partial \bar{H}_i}{\partial \theta_i} = (n-1) \frac{\cos \theta_i}{\sin \theta_i} \quad (3.11)$$

$$B_{ij}(\theta, \phi) = -\frac{1}{n} \frac{\partial \bar{H}_{ij}}{\partial \theta_i} = \frac{\frac{\partial C}{\partial \theta_i}(\theta_i, \theta_j) + \frac{\partial D}{\partial \theta_i}(\theta_i, \theta_j) \cos n\phi_{ij}}{C(\theta_i, \theta_j) + D(\theta_i, \theta_j) \cos n\phi_{ij}} \quad i \neq j. \quad (3.12)$$

PROOF The principle of symmetric criticality implies that the relative equilibria of this form are the solutions of the equations

$$\frac{\partial H}{\partial \theta_i} = \frac{\partial \Phi_\xi}{\partial \theta_i} \quad \frac{\partial H}{\partial \phi_i} = \frac{\partial \Phi_\xi}{\partial \phi_i} \quad i = 1, \dots, k$$

where

$$\Phi_\xi = \xi \mu = \xi \left( \lambda_n - \lambda_s + \sum_{j=1}^k \lambda_j \cos \theta_j \right).$$

A straightforward computation gives the equations in the statement of the proposition.  $\square$

Note that

$$C(\theta_j, \theta_i) = C(\theta_i, \theta_j) \quad D(\theta_j, \theta_i) = D(\theta_i, \theta_j)$$

and so

$$A_{ji}(\theta, \phi) = -A_{ij}(\theta, \phi)$$

while  $B_{ji}(\theta, \phi)$  is equal to  $B_{ij}(\theta, \phi)$  with  $\theta_i$  and  $\theta_j$  interchanged.

### 3.1.1 A single ring

A configuration consisting of a single regular  $n$ -ring of identical vortices together with  $k_p = 0, 1$  or  $2$  ‘polar’ vortices lies in the stratum  $\mathbf{C}_{nv}(R, k_pp)$ .

**Proposition 3.4** *A configuration consisting of a ring of  $n$  vortices with vorticity  $\lambda \neq 0$  and polar vortices with vorticities  $\lambda_n$  and  $\lambda_s$  is always a relative equilibrium. The angular velocity and centre of vorticity of a ring with co-latitude  $\theta$  are given by*

$$\xi = \frac{1}{\sin^2 \theta} \{ (n-1)\lambda \cos \theta + [(\lambda_n - \lambda_s) + (\lambda_n + \lambda_s) \cos \theta] \} \quad (3.13)$$

$$\mu = n\lambda \cos \theta + \lambda_n - \lambda_s. \quad (3.14)$$

A glance at these formulae shows that in general equilibria can occur for  $\mu \neq 0$  and non-equilibrium relative equilibria for  $\mu = 0$ . This is not the case for Hamiltonian systems of the form ‘kinetic + potential’. For systems with  $\mathbf{SO}(3)$  symmetry, (relative) equilibria with at most one of  $\mu$  and  $\xi$  vanishing are called ‘transversal’ in [36].

**PROOF** The expression for  $\mu$  is a special case of equations (3.1). Since  $k = 1$  and  $A_{11} = B_{11} = 0$  equation (3.7) is trivially satisfied while (3.8) can be solved for  $\xi$ .  $\square$

**Corollary 3.5** *For a system of  $n$  vortices with vorticity  $\lambda$ , together with up to two further vortices with vorticities  $\lambda_n$  and  $\lambda_s$ :*

1. *There exists a unique  $\mathbf{C}_{nv}(R, k_pp)$  relative equilibrium for each  $\mu$  with  $|\mu - (\lambda_n - \lambda_s)| < n|\lambda|$ ;*
2. *The  $\mathbf{C}_{nv}(R, k_pp)$  equilibria ( $\xi = 0$ ) satisfy  $\mu = 0$  if and only if either  $\lambda_n = \lambda_s$  or  $\lambda = \lambda_n + \lambda_s$ .*

Dritchell and Polvani [38] consider the stability of a single ring of identical vortices without polar vortices. They show that the ring  $\mathbf{C}_{nv}(R)$  is *linearly stable* when the co-latitude  $\theta$  satisfies:

$n$	range of stability	$n$	range of stability
3	all $\theta$	4	$\cos^2 \theta > 1/3$
5	$\cos^2 \theta > 1/2$	6	$\cos^2 \theta > 4/5$

while for  $n > 6$  the ring is never stable. In a forthcoming paper [24] we show that the linearly stable rings are in fact Lyapounov stable and we determine the stabilities of other symmetric relative equilibria, including those of type  $\mathbf{C}_{nv}(R, p)$  and  $\mathbf{C}_{nv}(R, 2p)$ .

### 3.1.2 Two rings

**Proposition 3.6** *A relative equilibrium consisting of two regular  $n$ -rings, each of identical vortices with non-zero vorticities, together with  $k_p = 0, 1$  or  $2$  ‘polar’ vortices, must lie in the closure of one of the strata  $\mathbf{C}_{nv}(2R, k_{pp})$  or  $\mathbf{C}_{nv}(R, R', k_{pp})$ . If either  $\xi \neq 0$  or  $\mu \neq 0$  then it must lie strictly in one these strata.*

Thus the two rings must be ‘aligned’ or ‘staggered’.

PROOF If  $k = 2$  and  $\lambda_1$  and  $\lambda_2$  are non-zero, then equations (3.7) reduce to

$$A_{12}(\theta_1, \theta_2, \phi_{12}) = \frac{D(\theta_1, \theta_2) \sin n\phi_{12}}{C(\theta_1, \theta_2) + D(\theta_1, \theta_2) \cos n\phi_{12}} = 0.$$

Since  $D(\theta_1, \theta_2)$  is never zero this holds if and only if  $\phi_{12}$  is an integer multiple of  $\pi/n$ . Even multiples give aligned rings, and hence relative equilibria in the closure of  $\mathbf{C}_{nv}(2R, k_{pp})$ , while odd multiples give staggered rings in the closure of  $\mathbf{C}_{nv}(R, R', k_{pp})$ .

All the strata in the frontiers of  $\mathbf{C}_{nv}(2R, k_{pp})$  and  $\mathbf{C}_{nv}(R, R', k_{pp})$  are zero-momentum strata. It therefore follows from Propositions 1.14 and 2.6 that if either  $\xi \neq 0$  or  $\mu \neq 0$  then the relative equilibrium must be strictly in the  $\mathbf{C}_{nv}(2R, k_{pp})$  and  $\mathbf{C}_{nv}(R, R', k_{pp})$  strata themselves.  $\square$

**Corollary 3.7** *Strata of types  $\mathbf{C}_n(2R, k_{pp})$ ,  $\mathbf{C}_{nv}(R_s, k_{pp})$  and  $\mathbf{D}_n(R, k_{pp})$  cannot contain relative equilibria.*

PROOF Points in the strata  $\mathbf{C}_n(2R, k_{pp})$  and  $\mathbf{C}_{nv}(R_s, k_{pp})$  consist of two rings and  $k_p$  poles. By Proposition 3.6 relative equilibria of this form must lie in the closure of  $\mathbf{C}_{nv}(2R, k_{pp})$  or  $\mathbf{C}_{nv}(R, R', k_{pp})$ . However the intersections of these closures with  $\mathbf{C}_n(2R, k_{pp})$  and  $\mathbf{C}_{nv}(R_s, k_{pp})$  are empty, giving the result.  $\square$

For systems of two aligned or staggered rings the equations (3.7, 3.8) reduce to

$$(n-1) \frac{\cos \theta_1}{\sin \theta_1} \lambda_1 + B_{12}(\theta_1, \theta_2) \lambda_2 = \xi \sin \theta_1 - \lambda_n \frac{\sin \theta_1}{1 - \cos \theta_1} + \lambda_s \frac{\sin \theta_1}{1 + \cos \theta_1} \quad (3.15)$$

$$B_{21}(\theta_1, \theta_2) \lambda_1 + (n-1) \frac{\cos \theta_2}{\sin \theta_2} \lambda_2 = \xi \sin \theta_2 - \lambda_n \frac{\sin \theta_2}{1 - \cos \theta_2} + \lambda_s \frac{\sin \theta_2}{1 + \cos \theta_2} \quad (3.16)$$

where  $B_{12}(\theta_1, \theta_2) = B_{21}(\theta_2, \theta_1)$  is the derivative of  $\log(C(\theta_1, \theta_2) + D(\theta_1, \theta_2))$  and  $\log(C(\theta_1, \theta_2) - D(\theta_1, \theta_2))$  with respect to  $\theta_1$  in the aligned and staggered cases respectively. Expressions for  $C + D$  and  $C - D$  are given in equations (3.5, 3.6). The next two paragraphs contain brief discussions of what we know about the solutions to these equations in the aligned and staggered cases, respectively.

**Aligned rings** The strata  $\mathbf{C}_{nv}(2R, k_{pp})$  have non-empty intersections with  $\Phi^{-1}(\mu)$  if and only if

$$\lambda_n - \lambda_s - n(|\lambda_1| + |\lambda_2|) < \mu < \lambda_n - \lambda_s + n(|\lambda_1| + |\lambda_2|). \quad (3.17)$$

If  $\lambda_1 \neq \lambda_2$  the quotients of these intersections in  $\mathcal{P}/\widehat{G}$  each have two components, corresponding respectively to  $\theta_1 < \theta_2$  and  $\theta_1 > \theta_2$ . If  $\lambda_1 = \lambda_2$  then the extra permutational symmetries result in connected quotient strata.

If  $\lambda_1$  and  $\lambda_2$  have the same sign then Theorem 2.5 implies that there must be at least two  $\widehat{G}$  orbits of relative equilibria for each allowed value of  $\mu$  when  $\lambda_1 \neq \lambda_2$ , and one when  $\lambda_1 = \lambda_2$ . As  $\mu$  approaches

the upper and lower limits both rings must converge to the same pole. Numerical investigations (using MAPLE) of equations (3.15, 3.16) for a number of cases suggest that the number of  $\widehat{G}$  orbits of relative equilibria that occur is in fact this minimum. It may be possible to prove this by a convexity argument similar to that in Theorem 3.8 below. However we haven't attempted to do this. Refer to examples in §4.

If  $\lambda_1$  and  $\lambda_2$  have opposite signs then we do not have an existence theorem and the analysis of specific examples shows that there can be values of  $\mu$  for which  $\mathbf{C}_{nv}(2R, k_pp)$  has a non-empty intersection with  $\Phi^{-1}(\mu)$ , but this intersection does not contain any relative equilibria. Numerical results suggest that relative equilibria do exist for  $\mu$  near to the upper and lower limits given by (3.17). As these limits are approached the rings must converge to opposite poles. Further investigations of equations (3.15, 3.16) are needed to determine the precise range of  $\mu$  values for which relative equilibria occur.

**Staggered rings** The strata  $\mathbf{C}_{nv}(R, R', k_pp)$  also have non-empty intersections with  $\Phi^{-1}(\mu)$  if and only if  $\mu$  satisfies (3.17). However in this case the quotients are always connected and so, when  $\lambda_1$  and  $\lambda_2$  have the same sign, Theorem 2.5 only implies the existence of a single  $\widehat{G}$  orbit of relative equilibria for each allowed value of  $\mu$ . Moreover these relative equilibria are only guaranteed to lie in the *closure* of  $\mathbf{C}_{nv}(R, R', k_pp)$ . If  $\lambda_1 = \lambda_2$  then this closure contains  $\mathbf{C}_{2nv}(R, k_pp)$ , ie single ring configurations with  $\theta_1 = \theta_2$ . This stratum must contain a relative equilibrium and so there may not be any in  $\mathbf{C}_{nv}(R, R', k_pp)$  itself. If  $\lambda_1 \neq \lambda_2$  then  $\mathbf{C}_{nv}(R, R', k_pp)$  is minimal and so will contain relative equilibria.

Numerical investigations using MAPLE suggest that if  $\lambda_1 = \lambda_2$  and there are no polar vortices then in addition to the single ring  $\theta_1 = \theta_2$  relative equilibria there is another family with  $\theta_1 \neq \theta_2$  which exists for  $\mu \in (-\mu_0, \mu_0)$  for some  $0 < \mu_0 < n(|\lambda_1| + |\lambda_2|)$ . As  $\mu$  approaches  $\pm\mu_0$  these relative equilibria bifurcate from the single  $2n$ -ring relative equilibria. Specific examples include the following pitchfork bifurcations:  $\mathbf{C}_{4v}(R)$  to  $\mathbf{C}_{2v}(R, R')$  in §4.2,  $\mathbf{C}_{4v}(R, p)$  to  $\mathbf{C}_{2v}(R, R', p)$  in §4.3 and the  $\mathbf{C}_{6v}(R)$  to  $\mathbf{C}_{3v}(R, R')$  in §4.4.

This behaviour persists in the presence of sufficiently small polar vorticities, as shown by the behaviour of the  $\mathbf{C}_{2v}(R, R', p)$  branch in §4.3 check. However as the polar vorticities increase the interval of  $\mu$  values for which the  $\mathbf{C}_{nv}(R, R', k_pp)$  relative equilibria exist shrinks and eventually disappears. Thus it appears that polar vorticity tends to suppress staggered pairs of rings.

An idea of the behaviour that is seen when  $\lambda_1 \neq \lambda_2$ , but both have the same sign, can be seen by considering small perturbations from the  $\lambda_1 = \lambda_2$  case. If the polar vorticities are not too large then perturbed pitchfork bifurcation will appear and there must be a range of  $\mu$  values for which there are multiple branches of relative equilibria of type  $\mathbf{C}_{nv}(R, R', k_pp)$ . The end points of these ranges will be given by fold bifurcations. As the polar vorticities increases the branch will straighten out until only a unique relative equilibrium is left for each value of  $\mu$  allowed by (3.17).

The case of two staggered rings with vorticities of opposite signs is even more complex. If  $\lambda_1 = -\lambda_2$  and  $\lambda_n = -\lambda_s$  then it is easily seen from equations (3.15, 3.16) that there is a branch of relative equilibria with  $\theta_1 = \pi - \theta_2$  which exists throughout the whole range of  $\mu$  values given by (3.17). Numerical evidence suggests that in addition, if the polar vorticities are not too large, there is a pair of pitchfork bifurcations from this branch to families of staggered relative equilibria with  $\theta_1$  and  $\theta_2$  closer to the diagonal  $\theta_1 = \theta_2$ . At higher polar vorticities one of these pitchfork bifurcations disappears and the other produces an almost vertical branch of relative equilibria along which the two rings converge to the same pole. Perturbing away from  $\lambda_1 = -\lambda_2$ ,  $\lambda_n = -\lambda_s$  yields fold bifurcations coming from the perturbed pitchforks and also values of  $\mu$  for which no staggered relative equilibria exist. A more detailed analysis of equations (3.15, 3.16) is needed to understand these better.

### 3.2 Equatorial equilibria

In this section, we show that provided all the vortices have the same sign vorticity, then for each cyclic ordering of the vortices there is a unique equilibrium configuration with all vortices lying on the equator. The uniqueness is, of course, modulo rotations. The fact that such equilibria exist is an instance of Theorem 2.2(2), since the subset of phase space where all vortices lie on the equator is the fixed point stratum  $C_h(NE)$ . The uniqueness depends on a convexity argument. A similar result may be true for relative equilibria, but we do not know.

The case when all the vorticities are equal suggests that for  $N > 3$  these configurations are always linearly unstable [38, 24].

**Theorem 3.8** *Let the  $N$  vortices be of positive vorticity. Then there is a unique equilibrium point  $\mathbf{x} = (x_1, \dots, x_N)$  such that all the  $x_i$  lie on the equator,  $x_1 = (1, 0, 0)$  and the points  $\{x_1, \dots, x_N\}$  appear in cyclic order around the equator.*

**PROOF** In the notation of Table 2, all vortices lying on a great circle corresponds to a point in phase space of type  $C_h(NE)$ . If, as in the statement of the theorem, we choose the great circle to be the equator, then the set of such configurations is parametrized by  $(\phi_1, \dots, \phi_N)$ , with each  $\phi_j \in [0, 2\pi]$  and  $\phi_i \neq \phi_j$ . We are therefore looking for critical points of the restriction of the Hamiltonian to this stratum. Note that fixing the great circle to be the equator restricts the orbits of the  $\mathbf{SO}(3)$  action to  $\mathbf{SO}(2)$  orbits.

Since all the points lie on the equator, we have

$$H(\phi_1, \dots, \phi_N) = - \sum_{i < j}^N \lambda_i \lambda_j \log(1 - \cos(\phi_i - \phi_j)).$$

The second derivative of this function is given by

$$H_{jj} = \lambda_j \sum_{i|i \neq j} \frac{\lambda_i}{1 - \cos(\phi_i - \phi_j)},$$

$$H_{ij} = - \frac{\lambda_i \lambda_j}{1 - \cos(\phi_i - \phi_j)} \quad (i \neq j),$$

where  $H_{ij} = \frac{\partial^2 H}{\partial \phi_i \partial \phi_j}$ . Note that  $H_{jj} > 0$  and  $H_{ij} < 0$  (for  $i \neq j$ ). Moreover the sum of every row of the Hessian matrix  $d^2H = (H_{ij})$  is zero (a consequence of the rotational symmetry) so it follows from Gerschgorin's theorem that all the eigenvalues are positive or zero.

Furthermore, zero is an eigenvalue of multiplicity precisely 1. For suppose  $\mathbf{u} \in \mathbf{R}^N$  is an eigenvector in the kernel of  $d^2H$  and  $i$  is such that (after possibly rescaling  $\mathbf{u}$ )

$$u_i = 1 \quad \text{and} \quad |u_j| \leq 1, \quad \forall j.$$

Then calculating the  $i^{\text{th}}$  row of the equation  $d^2H\mathbf{u} = 0$  gives

$$H_{ii} + \sum_{j \neq i} H_{ij} u_j = 0,$$

which implies that the  $u_j = 1$  too. Thus  $\mathbf{u} = (1, 1, \dots, 1)$ .

It follows that when restricted to any transversal to the  $\mathbf{SO}(2)$ -orbit, for example  $\phi_1 = 0$ , the Hessian is a locally convex function. In each connected component of the domain of definition, the function therefore has precisely one critical point, which is in fact a local minimum. The different connected components correspond to the different cyclic orderings of the vortices.  $\square$

## 4 Examples

In this section, we consider the existence of symmetric relative equilibria in the special cases of  $N$  identical vortices, with  $N = 3, 4, 5$  and  $6$ . Without loss of generality we suppose  $\lambda_j = 1$  for  $j = 1, \dots, N$ . For  $N = 3$ , the system is a special case of that studied by Kidambi and Newton [14] and by Pekarsky and Marsden [37], whose results show that the relative equilibria we give below are the only ones. In the other cases, results due to Kirwan [19] on the topology of the reduced phase spaces show that the relative equilibria listed below do not satisfy the Morse inequalities, so that there must exist other relative equilibria.

**Summary of non-existence results** The only general non-existence results, valid for all  $N$ , were found in Corollary 3.7, where it was shown that there are no relative equilibria with  $\widehat{G}$  symmetry types  $\mathbf{C}_n(2R, \ell p)$ ,  $\mathbf{C}_{nv}(R_s, \ell p)$ , or  $\mathbf{D}_n(R, \ell p)$  but that the two rings must be aligned or staggered (types  $\mathbf{C}_{nv}(2R)$  or  $\mathbf{D}_{nh}(R)$  and  $\mathbf{C}_{nv}(R, R')$  or  $\mathbf{D}_{nd}(R)$  respectively). In [14] it is shown that the only relative equilibria for  $N = 3$  identical vortices are those of types  $\mathbf{C}_{3v}(R)$  and  $\mathbf{C}_{2v}(R, p)$ . Further results can be obtained by direct algebraic computation on a case-by-case approach. For example, for  $N = 4$ , in the 1-dimensional fixed point space corresponding to  $\mathbf{D}_{2h}(R)$  it is an easy computation to show that the only critical points are those with symmetry  $\mathbf{D}_{4h}(R)$ . Consequently there are no equilibria with symmetry  $\mathbf{D}_{2h}(R)$ , and no relative equilibria either, as  $\mathbf{D}_{2h}(R)$  is a zero-momentum stratum — see Proposition 2.6. Similar arguments apply to strata of types  $\mathbf{C}_{2v}(R, R', 2p)$  (with  $N = 6$ )

**Description of tables and figures** Most of the results depicted in the tables and figures below are obtained algebraically, by solving the equations for relative equilibria. However, for strata of dimension greater than 2, when the algebraic manipulations are not feasible, we use numerical methods with MAPLE. Colour pictures and some of the Maple files are available on the web, at

<http://www.inln.cnrs.fr/~montaldi/Research/Vortices>

The column headed *multiplicity* refers to the number of inequivalent relative equilibria that exist in a given stratum for each value of  $\mu$ .

multiplicity is in fact lower bound in some cases??

### 4.1 3 identical vortices

For  $N = 3$  there is a unique equilibrium, which has symmetry  $\mathbf{D}_{3h}(r)$  (an equilateral triangle), and for  $\mu \in (0, 3)$  there are relative equilibria with symmetries as follows:

Symmetry type	domain of existence	limit as $\mu \rightarrow 0$	multiplicity
$\mathbf{C}_{3v}(R)$	$\mu \in (0, 3)$	$\mathbf{D}_{3h}(r)$	unique
$\mathbf{C}_{2v}(R, p)$	$\mu \in (0, 3)$	$\mathbf{D}_{3h}(r)$	two if $\mu \in (0, 1)$

Recall that configurations of point vortices of type  $\mathbf{C}_{3v}(R)$  consist of 3 points arranged in an equilateral triangle around a fixed latitude. The motion of such a relative equilibrium is just rigid rotation around that latitude. The configuration  $\mathbf{C}_{2v}(R, p)$  consists of a polar vortex, and a pair (a “2-ring”) on a fixed latitude and on opposite longitudes. The motion is rigid rotation about the axis through the polar vortex. Note that if the co-latitude is  $\theta$  then  $|\Phi| = |1 + 2 \cos \theta|$ , so that for each value  $\mu \in (0, 1)$  of  $|\Phi|$  there are two possible latitudes, corresponding to a long or acute isosceles triangle, and a flat, or obtuse isosceles triangle. This explains the multiplicity 2 given in the fourth column.

Figure 3: The energy momentum diagram for all relative equilibria for 3 identical vortices

### 4.2 4 identical vortices

For  $N = 4$  there are two equilibria. For one the vortices form a square on the equator, with symmetry  $\mathbf{D}_{4h}(r)$ , and for the other they form the vertices of a tetrahedron, with symmetry  $\mathbb{T}_d(v)$ . Both of these have centre of vorticity  $\Phi = 0$ . For  $\mu \in (0, 4)$  there are relative equilibria with  $|\Phi| = \mu$  of types given in the following table:

Symmetry type	domain of existence	limit as $\mu \rightarrow 0$	multiplicity
$\mathbf{C}_{4v}(R)$	$\mu \in (0, 4)$	$\mathbf{D}_{4h}(r)$	unique
$\mathbf{C}_{3v}(R, p)$	$\mu \in (0, 4)$	$\mathbb{T}_d(v)$	two if $\mu \in (0, 2)$
$\mathbf{C}_{2v}(R, 2p)$	$\mu \in (0, 2)$	$\mathbf{D}_{4h}(r)$	unique
$\mathbf{C}_{2v}(2R)$	$\mu \in (0, 4)$	$\mathbf{D}_{4h}(r)$	unique
$\mathbf{C}_{2v}(R, R')$	$\mu \in (0, 4\sqrt{3}/3)$	$\mathbb{T}_d(v)$	unique

Recall that  $\mathbf{C}_{2v}(R, 2p)$  consists of two polar vortices with two more on the same latitude and opposite longitudes,  $\mathbf{C}_{2v}(2R)$  consists of two aligned 2-rings and  $\mathbf{C}_{2v}(R, R')$  consists of two staggered 2-rings. The non-uniqueness of the  $\mathbf{C}_{3v}(R, p)$  relative equilibria for  $\mu \in (0, 2)$  is due to the fact that ‘long’ and ‘flat’ configurations may have the same value of  $|\Phi|$ .

The first four entries in the table are minimal  $\widehat{G}$ -strata (see Table 4), while the occurrence of the last one is predicted by studying bifurcations from the  $\mathbb{T}_d(v)$  equilibrium. The precise domain of existence is found by solving the equation for relative equilibria within the two-dimensional stratum  $\mathbf{C}_{2v}(R, R')$ . This computation also shows that as  $\mu \rightarrow 4\sqrt{3}/3$ , the relative equilibria of type  $\mathbf{C}_{2v}(R, R')$  approach those of type  $\mathbf{C}_{4v}(R)$ . This value of  $\mu$  is the same as that for which the  $\mathbf{C}_{4v}(R, p)$  relative equilibrium loses stability, as described in §3.1.1.

Figure 4: The energy momentum diagram for the symmetric relative equilibria for 4 identical vortices

### 4.3 5 identical vortices

For  $N = 5$  there are three types of symmetric equilibrium: a regular pentagon on the equator  $\mathbf{D}_{5h}(r)$ , a non-equatorial square with a pole  $\mathbf{C}_{4v}(R, p)$  and an equatorial equilateral triangle with two poles  $\mathbf{D}_{3h}(r, p)$ . For  $\mu \in (0, 5)$  there are symmetric relative equilibria with  $|\Phi| = \mu$  of types given in the following table:

Symmetry type	domain of existence	limit as $\mu \rightarrow 0$	multiplicity
$\mathbf{C}_{5v}(R)$	$\mu \in (0, 5)$	$\mathbf{D}_{5h}(r)$	unique
$\mathbf{C}_{4v}(R, p)$	$\mu \in (0, 5)$	$\mathbf{C}_{4v}(R, p)$	two if $\mu \in (0, 3)$
$\mathbf{C}_{3v}(R, 2p)$	$\mu \in (0, 3)$	$\mathbf{D}_{3h}(r, p)$	unique
$\mathbf{C}_{2v}(2R, p)$	$\mu \in (0, 5)$	$\mathbf{D}_{5h}(r)$	two if $\mu \in (0, 3)$
$\mathbf{C}_{2v}(R, R', p)$	$\mu \in (0, 1)$	$\mathbf{D}_{3h}(r, p)$	two
$\mathbf{C}_h(2R, E)$	$\mu \in (0, \mu_1)$	$\mathbf{C}_{4v}(R, p)$	unique
$\mathbf{C}_h(R, 3E)$	$\mu \in (0, \mu_2)$	$\mathbf{C}_{4v}(R, p)$	unique

$\mathbf{C}_{2v}(R, R', p)$  Both branches of this family of relative equilibria bifurcate from the respective branches of  $\mathbf{C}_{4v}(R, p)$  at  $\mu = 1$ .

$\mathbf{C}_h(k_R R, k_E E)$  Numerical evidence (using MAPLE) suggests that  $\mu_1 = 5$  and  $\mu_2 = 3$  and that both the  $\mathbf{C}_h(2R, E)$  and  $\mathbf{C}_h(R, 3E)$  relative equilibria persist until they approach the diagonal. It also appears that the latter is asymptotic to the  $\mathbf{C}_{3v}(R, 2p)$  family.

### 4.4 6 identical vortices

For  $N = 6$  there are four types of symmetric equilibrium: the vertices of an octahedron  $\mathbb{O}_h(v)$ , a regular hexagon on the equator  $\mathbf{D}_{6h}(r)$ , a non-equatorial pentagon with a pole  $\mathbf{C}_{5v}(R, p)$ , and a ‘triangular prism’

Figure 5: The energy momentum diagram for the symmetric relative equilibria for 5 identical vortices

$\mathbf{D}_{3h}(R)$ . For  $\mu \in (0, 6)$  there are relative equilibria with  $|\Phi| = \mu$  of types given in the following table:

Symmetry type	domain of existence	limit as $\mu \rightarrow 0$	multiplicity
$\mathbf{C}_{6v}(R)$	$\mu \in (0, 6)$	$\mathbf{D}_{6h}(r)$	unique
$\mathbf{C}_{5v}(R, p)$	$\mu \in (0, 6)$	$\mathbf{C}_{5v}(R, p)$	two if $\mu \in (0, 4)$
$\mathbf{C}_{4v}(R, 2p)$	$\mu \in (0, 4)$	$\mathbb{O}_h(v)$	unique
$\mathbf{C}_{3v}(2R)$	$\mu \in (0, 6)$	$\mathbf{D}_{3h}(R)$	unique
$\mathbf{C}_{2v}(3R)$	$\mu \in (0, 6)$	$\mathbf{D}_{6h}(r)$	unique
$\mathbf{C}_{2v}(2R, R')$	$\mu \in (0, 6)$	$\mathbb{O}_h(v)$	unique
$\mathbf{C}_{2v}(2R, 2p)$	$\mu \in (0, 4)$	$\mathbf{D}_{6h}(r)$	unique
$\mathbf{C}_{3v}(R, R')$	$\mu \in (0, 2\sqrt{5}/5)$	$\mathbb{O}_h(v)$	unique
$\mathbf{C}_{2v}(R_s, R)$	$\mu \in (0, \mu_{2v})$	$\mathbf{D}_{3h}(R)$	two if $\mu$ small
$\mathbf{C}_h(2R, 2E)$	$\mu \in (0, \mu_h)$	$\mathbf{C}_{5v}(R, p)$	two if $\mu$ small

$\mathbf{C}_{3v}(R, R')$  As  $\mu \rightarrow 2\sqrt{5}/5$  this family approaches the  $\mathbf{C}_{6v}(R)$  family.

$\mathbf{C}_h(2R, E)$  The bifurcation result in Proposition 2.8 shows that there are 2 families of relative equilibria of this type bifurcating from the  $\mathbf{C}_{5v}(R, p)$  equilibrium at  $\mu = 0$ , whence the multiplicity 2 given in the table. Numerical evidence suggests that one of these families exists for all  $\mu \in (0, 6)$ , so that  $\mu_h = 6$ , and the other exists for  $\mu \in (0, 4)$ , and as  $\mu \rightarrow 4$  one vortex approaches one pole and the remaining 5 approach the other pole.

$\mathbf{C}_{2v}(R_s, R)$  Again the existence of this branch follows from the bifurcation result in Proposition 2.8, this time applied to the  $\mathbf{D}_{3h}(R)$  equilibrium. We have not attempted to follow this branch to obtain an estimate for  $\mu_{2v}$ .

It might be expected that for  $N = 6$  there are relative equilibria of type  $C_{2v}(R, R', 2p)$ , but in fact a simple algebraic calculation shows there are none.

Figure 6: The energy momentum diagram for the symmetric relative equilibria for 6 identical vortices

## Appendix: The finite subgroups of $O(3)$ and their orbit types

### The finite subgroups of $SO(3)$

$C_n$  This is the cyclic group of order  $n$  acting by rotations about an axis in  $\mathbf{R}^3$  which we will take to be the  $z$ -axis. There are two types of orbit for the restriction of this action to  $S^2$ , each of the two fixed points at the poles (type  $p$ ) and any horizontal regular  $n$ -gon (type  $R$ ). These regular  $n$ -gons we call  $n$ -rings.

$D_n$  This is the dihedral group of order  $2n$  consisting of the cyclic subgroup  $C_n$  described above together with  $n$  rotations by  $\pi$  about axes which lie in the  $(x, y)$ -plane. The orbits in  $S^2$  are the two poles, forming a single orbit ( $p$ ), an ‘equatorial’ regular  $n$ -gon in the  $(x, y)$  plane ( $r$ ), its dual (also  $r$ ) and the single orbits formed by any two regular  $n$ -gons placed at opposite latitudes ( $R$ ).

$T$  This is the group of orientation preserving symmetries of a regular tetrahedron in  $\mathbf{R}^3$ . It has three different types of orbit in  $S^2$ . One ( $v$ ) consists of the two orbits given by the set of vertices of the tetrahedron and those of the dual tetrahedron, or equivalently the mid-points of the faces of the original tetrahedron. The next type ( $e$ ) is the single orbit of the mid-points of the edges of the tetrahedron. Finally there are the regular orbits ( $R$ ) given by generic points in  $S^2$ .

$O, I$   $O$  is the group of orientation preserving symmetries of the octahedron while  $I$  is the group of orientation preserving symmetries of the icosahedron. Each group has four orbit types, those corresponding to the vertices of the polyhedron ( $v$ ), those corresponding to the mid-points of its faces ( $f$ ), or equivalently the vertices of the dual cube or dodecahedron, those given by the mid-points of the edges of the polyhedron ( $e$ ), and the regular orbits ( $R$ ).

### The other finite subgroups of $O(3)$

$C_{nv}$  The dihedral group of order  $2n$  consisting of  $C_n$  together with reflections in  $n$  equally angled planes containing the rotation axis of  $C_n$ , which we continue to take as the  $z$ -axis. The orbits for the action of this group on  $S^2$  are the poles (two single point orbits), regular horizontal  $n$ -gons and their duals (separate orbits with  $n$  points in each), and generic orbits, taking the form of horizontal semi-regular  $2n$ -gons. Note that there is a difference between  $n$  even and  $n$  odd: the symmetry groups of a regular  $n$ -gon and its dual

always coincide, however their actions are not the same on the set of vertices if  $n$  is even, while if  $n$  is odd they do coincide (up to relabelling).

$C_{nh}$  The cyclic group of order  $2n$  generated by the elements of  $C_n$  together with reflection in the  $(x, y)$ -plane. The possible orbits are the single orbit formed by the two poles, equatorial regular  $n$ -gons, and pairs of vertically aligned  $n$ -gons at opposite latitudes.

$D_{nh}$  The dihedral group of order  $4n$  obtained by combining  $C_{nv}$  and  $C_{nh}$ . It also contains the  $SO(3)$  subgroup  $D_n$ . The orbits in  $S^2$  are the two poles (forming a single orbit), an ‘equatorial’ regular  $n$ -gon in the  $(x, y)$  plane and its dual (two separate orbits), vertically aligned pairs of  $n$ -gons at opposite latitudes, equatorial semi-regular  $2n$ -gons, and vertically aligned pairs of semi-regular  $2n$ -gons at opposite latitudes.

$D_{nd}$  The group of order  $4n$  generated by the dihedral group  $D_n$  together with  $n$  reflections in vertical planes bisecting the 2-fold rotation axes of  $D_n$ . The orbits in  $S^2$  are the two poles (a single orbit), an equatorial regular  $2n$ -gon, staggered pairs of regular  $n$ -gons, and staggered pairs of semi-regular  $2n$ -gons. Here ‘staggered’ means that the  $n$ -gons or  $2n$ -gons are at opposite latitudes and offset by  $\pi/n$  relative to each other. In this case the staggered configurations include equatorial configurations as special cases.

$S_{2n}$  The index two subgroup of  $C_{2nh}$  generated by rotation by  $\pi/n$  composed with reflection in the  $(x, y)$ -plane. Its orbits consist of the two poles, forming a single orbit, and vertically staggered pairs of regular  $n$ -gons at opposite latitudes.

$C_h$  The order two group generated by reflection in the  $(x, y)$ -plane. Orbits can consist of either a single point in the fixed point set (equator), or a pair of points on the same longitude and at opposite latitudes.

$C_i$  The order two group generated by the antipodal map of  $S^2$ . It acts freely, so every orbit consists of two antipodal points.

$\mathbb{T}_d$  The group of all rotational and reflectional symmetries of a regular tetrahedron. The orbits on  $S^2$  are the vertices of the tetrahedron or its dual (two orbits with 4 points in each), the mid-points of the tetrahedron (forming the vertices of an octahedron), the orbits generated by a generic point on an edge of a tetrahedron (12 points per orbit), and regular  $\mathbb{T}_d$  orbits.

$\mathbb{T}_h$  The group generated by  $\mathbb{T}$  and the antipodal map on  $S^2$ . It contains reflections in 3 orthogonal planes which we may take to be the coordinate planes. We call the intersections of the coordinate planes with  $S^2$  ‘equators’. The points where the coordinate axes meet  $S^2$  are then the mid-points of the edges of a tetrahedron and also the mid-points of the edges of its dual. The orbits of  $\mathbb{T}_h$  on  $S^2$  are the set of vertices of the tetrahedron and its dual (a single orbit with 8 points), the set of mid-points of the edges of the tetrahedron, orbits of generic points in equators (12 points per orbit), and regular  $\mathbb{T}_h$  orbits.

$\mathbb{O}_h$  The group of all rotational and reflectional symmetries of a regular octahedron. There are six different orbit types for the action on  $S^2$ : the vertices of the octahedron, the mid-points of the faces, the mid-points of the edges, orbits of generic points on face-bisectors, orbits of generic points on edges, and regular  $\mathbb{O}_h$  orbits.

$\mathbb{I}_h$  The group of all rotational and reflectional symmetries of a regular icosahedron. There are five different orbit types: the vertices of the icosahedron, the mid-points of the faces, the mid-points of the edges, orbits of generic points on edges, and regular  $\mathbb{I}_h$  orbits.

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