Exact solutions of a energy-enstrophy theory for the barotropic vorticity equation on a rotating sphere

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Abstract

The equilibrium statistical mechanics of the energy-enstrophy theory for the barotropic vorticity equation is solved exactly in the sense that a explicitly non-Gaussian configurational integral is calculated in closed form. A family of lattice vortex gas models for the barotropic vorticity equation (BVE) is derived and shown to have a well-defined nonextensive continuum limit as the coarse-graining is refined. This family of continuous-spin lattice Hamiltonians is shown to be nondegenerate under different point vortex discretizations of the BVE. Under the assumption that the energy and the enstrophy (mean squared absolute vorticity) are conserved, a long range version of Kac's Spherical Model with logarithmic interaction is derived and solved exactly in the zero total circulation or neutral vortex gas case by the method of steepest descent. The spherical model formulation is based on the fundamental observation that the conservation of enstrophy is mathematically equivalent to Kac's spherical constraint. Two new features of this spherical model are (i) it allows negative temperatures, and (ii) a nonextensive thermodynamic limit where the strength of the interaction scales with the number of lattice sites but where the size of the physical domain remains fixed; novel interpretations of the saddle point criterion for negative temperatures will be formulated. This Spherical model is shown to have a free energy that is analytic in the properly scaled inverse temperatures $\tilde{\beta}$ in the range $0 = \tilde{\beta}_* < \tilde{\beta} < \tilde{\beta}_c = \frac{N_*^2 \pi^2}{2K}$ in the nonextensive continuum limit, with K being the fixed value of the enstrophy. The boundary $\tilde{\beta}_* = 0$ agrees with the known numerical and analytical results on the occurrence of coherent or ordered structures at negative temperatures. Spin-spin correlations are calculated giving two-point vorticity correlations that are important to the study of turbulence. Physical interpretations of the results in this paper are obtained and applied to planetary atmospheres.

Keywords: Barotropic vorticity equation, statistical mechanics, spherical model, 2-D turbulence, inverse cascade

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http://online.itp.ucsb.edu/online/hydrot00/si-fluidx.html

1 Introduction

The equilibrium statistical mechanics of the planar vortex gas originated with the work of Onsager [46]. He initiated the discussion on negative temperature states for planar vortex systems, in which coherent structures consisting of large clusters of like-signed vortices occur. Onsager proposed that the coherent structures gave evidence of a inverse cascade of energy in 2-D ideal flows [1], [30], [27]. We will call this theory, the Onsager vortex gas theory. Onsager himself commented that there is an inherent ambiguity in his theory in the choice of point vortex representation for a given continuous vorticity field. We will return to this point later.

Onsager's work was followed by several interesting papers in the physics literature [44], [38] giving heuristic derivations of a mean field theory which has become known as the Onsager-Joyce-Montgomery (OJM) theory. Eyink and Spohn [20] proved rigorously, in the case of a one species planar vortex gas, that the entropy in a nonextensive continuum limit, has a maximum at a certain energy, which proved that negative temperature states exist. Exact solutions of the Onsager mean field equations obtained by Ting, Chen and Lee [57] confirm the existence of coherent structures. Recent numerical results cited in [6], [51] provide evidence of coherent structures in 2-D flows. Much of these results have been extended to the guiding center model of plasmas in view of the similarity in mathematical form between the planar point vortex problem and plasma problem [28]. We distinguish this OJM mean field vortex gas theory from the above Onsager theory. The OJM theory is a mean field lattice vortex gas theory while the Onsager theory is a full fledged vortex gas theory. The OJM theory like its parent Onsager theory suffers from a degeneracy arising from the choice of a particular point vortex discretization of the Euler vorticity field. We will return to the discussion of more recent developments of mean field theories for the Euler equation.

In the early fifties Batchelor [1] and Lee [29] developed a spectral representation for 3-D ideal fluids in order to further study the statistical dynamics of turbulence in fluids and plasmas. In particular, Lee [29] proved a Liouville theorem for the 3-D Euler flow in spectral coordinates. In 2-D it is known that any continuous function of vorticity is a conserved quantity of the Euler flow. Later Kraichnan [27] introduced a truncated version of the spectral formulation and showed that in 2-D, the energy and enstrophy remain conserved quantities under the truncated spectral dynamics. Because of the conservation of energy and enstrophy, it was argued in [1] that there is a 2-D inverse cascade in energy through an inertial range, and also a forward cascade in enstrophy through a different inertial range. The 1975 paper of Kraichnan [27] contains two probability measures, one is canonical in both energy and enstrophy, and the other is micro-canonical in both energy and enstrophy. In the same paper, he presented a solution for the energy spectrum of 2-D turbulence in terms of the first of these probability measures, by arguing that since energy and enstrophy are quadratic forms, the partition function is a Gaussian integral which can be calculated exactly. Kraichnan's Gaussian solution is not physically meaningful in the sense that a *low temperature catastrophe (* apres Kac cf. [2], [56]) occurs and the Gaussian partition function is not defined below the critical temperature. On the other hand, the second probability measure which is micro-canonical in both energy and enstrophy is too difficult to derive analytically, and numerical results are discussed in [28]. Clearly, a physically meaningful analytical solution of the energy-enstrophy theory is highly desired.

The aim of this paper is twofold: (I) to solve exactly the energy-enstrophy theory for the equilibrium statistics of the barotropic vorticity equation [48], and (II) to formulate a Onsager lattice vortex gas theory of the barotropic vorticity equation that does not have the degeneracy mentioned above. The equilibrium statistics in a certain nonextensive continuum limit of the energy-enstrophy is closely related to special relaxation end-state solutions of the barotropic vorticity equations, which exhibit the phenomena of coherent structures and inverse cascades (cf. [1], [41], [27], [30] for 2-D flows). We stress here the fact that the three constraints in our approach are natural fluid dynamical invariants, namely, the energy, the total circulation and the enstrophy or mean-squared absolute vorticity. The zero total circulation problem (neutral vortex gas) is the most widely studied case in planar vortex dynamics because, in view of Stokes theorem, it corresponds to the planar problem of periodic flows on a square domain. For the barotropic vorticity equation on a sphere, the same arguments again tell us that the most natural case is indeed the neutral vortex gas problem.

A exact statistical mechanics solution for the Energy-Enstrophy theory developed by Batchelor [1], Lee [29] and Kraichnan [27] for planar ideal flows has recently been discovered by Lim [34], [35]. It is based on the following observations: (i) for periodic boundary conditions on a square domain, the Energy-Enstrophy model is equivalent to a three constraints equilibrium statistical mechanics lattice vortex gas model, (ii) this lattice vortex gas model is mathematically equivalent to the family of continuous-spin lattice models known as spherical models because the enstrophy constraint has the same mathematical structure as the spherical constraint [2], and (iii) the exact partition function of the spherical models give the equilibrium probability measure $P = \frac{1}{Z}\exp(-\beta E)\delta(\Omega - K)$ which is canonical in the energy E and micro-canonical or sharp in the enstrophy Ω which is constrained to take the fixed value K. The major difference between this probability and the Gaussian one in Kraichnan's paper [27] is that it remains well-defined for all temperatures although it can loose analyticity as a function of temperature at some isolated values of temperature. This last property confers an additional theoretical benefit in the sense that the values at which the spherical model's partition function is not an analytic function of temperature, are precisely the critical temperatures of a phase transition. These critical temperatures and the correlation functions will be obtained exactly.

Following Onsager and Kac, we emphasize the importance of exact solutions in tractable models of phase transitions because numerical solutions can only hint at the existence of phase transition, but cannot be used to prove it. This is due to the fact that any finite dimensional approximation of the full problem must have thermodynamic quantities that are analytic functions over the whole range of temperatures since its partition function must be polynomial. One must take a continuum or thermodynamic limit in the problem before any non-analytic behaviour in thermodynamic quantities will arise. Non-analytic behaviour over some part of the temperature regime is of course the hallmark of phase transitions. On the other hand, once a phase transition has been rigorously established in a problem, numerical calculations can be gainfully used to further one's understanding of the properties at phase transition.

Two new features of the spherical models derived in this paper are worth remarking here, namely, (i) negative temperatures and (ii) a nonextensive thermodynamic limit that is suitable for continuum (macroscopic) fluid mechanics. Both these features are used in the formulation of its critical properties. Negative temperatures in statistical mechanics are not new nor restricted to vortex statistical mechanics; we refer the reader to Ramsey's 1950 paper for a detailed discussion of negative temperatures in nuclear spin systems. The second new feature is unique to inviscid computational fluid dynamics and the statistical mechanics of vortical flows. In both, numerical computations and statistical mechanics of continuum fluid flows, one formulates a lattice approximation for a periodic flow domain say, and then refine the lattice as required or ad infinitum. This coarse-graining procedure is distinguished by the fact that the size of the physical domain remains fixed while the strength of the interaction in the lattice Hamiltonian decreases in inverse proportion to the number of lattice sites. The inverse temperature β in the problem will scale with the number of lattice sites.

Our usage of the spherical constraint here as well as in [34], [35] is not as a approximation to some physical quantity, but as a latttice analogue of a naturally conserved quantity of 2-D ideal fluids, known as enstrophy. Moreover, the problems we address here, on one hand, the energy-enstrophy theory of the barotropic vorticity equation, and on the other hand, the Onsager vortex gas, are classical theories that were developed by well-known theoretical physicists about fifty years ago. In the spirit of Onsager and Kac, we aim to prove rigorously the existence of a phase transition in the energy-enstrophy theory and the Onsager theory for the barotropic vorticity equation. As far as we know, all previous applications of the spherical model is based on the idea that it provides a reasonable approximation to physical models in condensed matter physics, that is moreover, solvable.

To provide some background, we will now give a very short summary of the most relevant literature. The Onsager theory is based on just the energy and total circulation constraints due to its vortex gas origins. The Energy-Enstrophy theory of Kraichnan, Lee, Leith and Batchelor is really based on three constraints in the case of periodic boundary conditions on a planar square domain. A similar theory for the barotropic vorticity equations on a rotating sphere was discussed by Holloway and Hendershott 1977, Salmon et al 1976, Herring 1977 (cf. review article [28]). Numerical simulations of the barotropic vorticity equation are reported in Tang and Orszag 1998, Rhines 1975, and Bretherton and Haidvogel 1976. Baroclinic two-layer models for the atmosphere have been analyzed by Welch and Tung and others (cf. the references in [58]).

Miller [42] and Robert and Sommeria [49], [50] independently discovered a equilibrium statistical theory for the planar Euler equation, which takes into account all its conserved quantitites. They derived lattice models which are not unlike our family of lattice vortex gas models, but they imposed an infinite number of constraints, and constructed a mean field theory. The infinite number of constraints make their theory difficult to apply because in many realistic problems such as those in geophysical flows, it is impossible to obtain data on more than a few constraints. Turkington [59] recently developed a few constraints mean field theory for the 2-D Euler equation, which has been succesfully applied by Majda et al [16], [17], [18] to quasi-geostrophic problems. See also the related articles by Carnevale and Frederiksen [7] on the effects of topography on the quasi-geostrophic energy-enstrophy models.

The question of how many constraints to include in a useful equilibrium statistical theory for ideal flows on the plane or the surface of a sphere, is an important one. In recent work, Majda and Holen [39] showed that few constraints theories such as the Onsager theory and the 2-D Energy-Enstrophy theory [29], [27], [28] are statistically sharp with respect to the infinite constraints theory, i.e., these theories and the Miller-Robert theory agree at low energies. Chorin [8] has earlier conjectured that a finite number of constraints is enough. But the question of exactly how many constraints are enough appears to be unanswered. By showing that three fundamental and natural constraints, namely energy, total circulation and enstrophy, lead to a exactly solvable problem, we have partly answered this question. The least number of constraints is obviously two (the Onsager theory), and three is definitely enough for a exact solution.

In the following sections we will derive a family of lattice vortex gas models for the barotropic vorticity equation, each member of which has a different number of constraints. We use a point vortex gas discretization of the barotropic vorticity equation [8]. The point vortex gas is essentially a sum of delta functions discretization of the vorticity field ω_a in the barotropic vorticity model, i.e.,

$$\omega_a(\vec{x}, t) = \sum_{j=1}^{N} \lambda_j \delta(\vec{x} - \vec{x}_j(t)). \tag{1}$$

In the limit of infinite N and $\lambda_j \to 0$, it can be shown that the dynamics of the vortex gas tends to the evolution of vorticity according to the barotropic vorticity equation [40]. Although our derivation relies on the point vortex discretizations of the barotropic vorticity equation (BVE), hence the label vortex gas, we stress that this family models the equilibrium statistical behaviour of the BVE and not some approximation of it. This is due to the fact that these lattice vortex gas models are studied under a proper refinement of the coarse-graining procedure which constitutes a nonextensive thermodynamic limit. In other words, as we refine the coarse-graining, the vortex strengths of the point vortices used in the discretization of the BVE are correspondingly reduced in inverse proportion to the total number of point vortices in the system. Meanwhile, the area of the flow domain (or physical extent of the lattice) is kept fixed at its actual or physical value. Under these assumptions, it is known that as the total number of point vortices increases to infinity, the vortex gas approximation of the BVE tends to the BVE in a well-defined rigorous manner [40].

Moreover, the lattice vortex gas models is a Eulerian approach which does not suffer from the highly singular behaviour of the point vortex gas (a Lagrangian type approach) which occurs when point vortices collide. This can be viewed in terms of the lattice providing a natural ultra-violet cutoff in wavenumber space [35]. We used the vortex gas discretization procedure to make contact with the Onsager theory. We showed in [35] that the lattice vortex gas models are mathematically equivalent to the truncated spectral models for 2-D turbulence [29], [27].

Under the assumption that the energy and enstrophy are conserved quantities, we will show that a particular member of this family is related to a exactly solvable Spherical Model Hamiltonian [2] with long range interaction and lattice dimensionality $d = 2$. We will use the Spherical Model formulation to compute a threshold temperature $T_* = \infty$ of the Onsager vortex gas and the energy-enstrophy model on the rotating sphere. This result agrees with both numerical and analytical work [33] indicating a qualitative change in the properties of the equilibrium state as the $T_* = \infty$ boundary is crossed. Specifically, the equilibrium states consist of a coherent structure (a central large vortex made up of many lattice sites with spins of the same signature) for negative temperatures, while for positive temperatures, the equilibrium states consist of a random mix of opposite sign vortices (spins). Indeed, this is the phenomena which Onsager [46] proposed as being indicative of a inverse cascade of energy from small to large scales in 2-D turbulence.

We will also compute a critical temperature T_c with $0 < T_c < \infty$ where the saddle point of the partition function Z_S of the spherical model sticks [2]. For temperatures outside of the analytic regime given by the high temperature range $T_c < T < \infty$, we will use another technique to compute the free energy and the two-point spin correlations in the spherical model [52]. The spin-spin correlations are directly related to the two-point vorticity correlations and indirectly related to the structure functions of turbulence theories. We refer the reader to the literature for more details on structure functions [8], [14], [13], [15], [22], [23], [47], [53].

We will also show that the lattice vortex gas model presented here, where the spin variables are local values of the vorticity, does not suffer from the degeneracy of the Onsager theory. Specifically, it is independent of the particular vortex gas discretization of the barotropic vorticity equations.

The physical interpretation of the mathematical results in this paper is an important part of the study of 2-D turbulence in ideal fluids on the sphere, and we devote a section at the end of this paper to that purpose.

2 Vortex gas for the barotropic vorticity equation

The barotropic vorticity equation [48] for flows on the surface of a rotating sphere is given by

$$\frac{D}{Dt}\omega_a = 0,$$

where ω_a is the absolute vorticity

$$\omega_a = \omega_r + \omega_p \tag{2}$$

which is the sum of the normal component ω_r of the relative vorticity and the normal component $\omega_p = 2\Lambda \cos\theta$ of the planetary vorticity, with Λ the rate of rotation of the sphere, and θ is the co-latitude measured from the north pole. The material derivative in the above barotropic equation is taken in the rotating frame, i.e.,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}_r \cdot \nabla,$$

where $\vec{u}_r$ is the relative velocity in the rotating frame. As shown in [48], the material derivatives of a scalar quantity such as ω_a, are equal in the fixed and rotating frames of reference. Thus, the barotropic vorticity equation can be treated equally as a statement of the conservation of absolute vorticity ω_a while following fluid elements in the fixed frame, that is,

$$\frac{D}{Dt}\omega_a = \left(\frac{\partial}{\partial t} + \vec{u}_a \cdot \nabla\right)\omega_a = 0, \tag{3}$$

where

$$\vec{u}_a = \vec{u}_r + (0, 2\Lambda \sin\theta)$$

is the absolute velocity of a fluid element in the fixed frame which is obtained by inverting the spherical Poisson equation

$$\Delta\Psi = -\omega_a$$

for the stream function Ψ. We will use equation (3) in the rest of this paper.

The problem of point vortices on a unit sphere is a well-known model and numerical approximation for the barotropic vorticity equation and is now used extensively in work on some aspects of atmospheric sciences and oceanography where the viscosity of the flow and the three-dimensionality of the fluid can be safely set aside. This set of conditions is known to be valid in many barotropic quasi-geostropic problems because of the large physical length scales and the relatively rapid rotations of the flows on a planetary domain [48], [45].

In the particle discretization of the absolute vorticity ω_a one writes

$$\omega_a(\mathbf{x},t) = \sum_{j=1}^{N} \lambda_j(t)\delta(\mathbf{x} - \mathbf{x}_j(t)), \tag{4}$$

where the point vortex strengths are denoted by λ_j (where it turns out that λ_j remain constant) and the positions of the vortices are given by $\mathbf{x}_j(t) \in S^2$. From (3), Bogomolov [4], and Kimura and Okamoto [25] derived a discrete Hamiltonian system which governs the dynamics of point vortices on a rotating sphere, with Hamiltonian function given by

$$\begin{aligned} H(x_1, \ldots, x_N) &= -\sum_{i<j} \lambda_i \lambda_j \log_e(1 - x_i \cdot x_j) \\ &= -\sum_{i<j} \lambda_i \lambda_j \log_e \sin(\eta_{ij}/2), \end{aligned} \tag{5}$$

in terms of the Green's function

$$G = -\frac{1}{2\pi} \log_e \sin(\eta_{12}/2) \tag{6}$$

for the problem

$$\Delta_{S^2} \Psi_a = -\omega_a$$

where Δ_{S^2} is the Laplacian on the surface of the unit sphere, and η_{12} is the angular separation between two points on the unit sphere S^2. In terms of the spherical coordinates (θ, φ) where θ is the co-latitude ($\theta = 0$ at the north pole) and φ is the longitude,

$$\cos \eta_{12} = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2).$$

We will need the following lattice eigenfunction expansion for the Green's function

$$G(\varphi_1 - \varphi_2, \theta_1, \theta_2) \sim \sum_{2\pi\vec{q}/L} \frac{1}{n(n+1)} Y_n^m(\varphi_1, \theta_1) Y_n^{m*}(\varphi_2, \theta_2) \tag{7}$$

where $2\pi\vec{q}/L = 2\pi(m, n)/L$ with $m \in \{-n, ..., -1, 0, 1, ..., n\}$ and $n \in \{0, 1, .., L-1\}$, and

$$\Delta_{S^2} Y_n^m + \frac{1}{n(n+1)} Y_n^m = 0.$$

The surface spherical harmonics

$$Y_n^m(\varphi, \theta) = e^{im\varphi} P_n^m(\cos\theta)$$

where $P_n^m(\cos\theta)$ are the associated Legendre polynomials, and

$$P_n^m(\cos\theta) = \sin^m(\theta) W_n^m(\cos\theta)$$

with $W_n^m(\cos\theta)$ a degree $(n-m)$ polynomial in $\cos\theta$.

The phase space P coincides with the cartesian product

$$P = (S^2)^N. \tag{8}$$

The symplectic structure on P is given by

$$\omega = \sum \lambda_j \pi_j^* \omega_{S^2},$$

where π_j is the Cartesian projection on to the jth factor, and ω_{S^2} is the natural symplectic form on S^2. The Hamiltonian vector field X_H is then given by

$$\dot{x_j} = X_H(\mathbf{x})_j = \sum_{i \neq j} \lambda_i \frac{x_i \times x_j}{1 - x_i . x_j}. \tag{9}$$

Besides the energy (or Hamiltonian itself), the integrals of motion correspond to the $SO(3)$ symmetry of the above Hamiltonian function, namely

$$\Phi = \sum_{j=1}^{N} \lambda_j x_j.$$

The reader is referred to the more recent work of Kidambi and Newton [24], Lim, Montaldi and Roberts [37], and the forthcoming book by Newton [45] for further results on the dynamics of point vortices on a sphere.

It must be stressed here that given an initial distribution of absolute vorticity $\omega_a^0(\mathbf{x})$, the discretization procedure in (4) gives the initial positions $\mathbf{x}_j(0)$ of the N point vortices after their strengths have been fixed; then, the point vortices move according to (9) and the positions $\mathbf{x}_j(t)$ of the vortices at some future time t determine approximately the absolute vorticity distribution $\omega_a(\mathbf{x},t)$, again through (4). In this way, (9) and (4) determine an approximation to the actual evolution of the absolute vorticity $\omega_a(\mathbf{x},t)$ under the barotropic vorticity equation (3). By comparing this problem with its planar counterpart, we expect that this numerical method will converge as the number of particles N tends to ∞ [40]. For the remainder of this paper, one should bear in mind that the Hamiltonian system (9) is indeed a model for the evolution of absolute vorticity.

3 Long Range Spherical Models for the Barotropic Vorticity equation

In this section we will first review the fundamental coarse-graining procedure for deriving lattice vortex gas models of two-dimensional turbulence [31], [34]. First we discuss the usual Onsager model which is based on point vortex occupation numbers (for each species of point vorices) at lattice sites. This Onsager theory suffers from the defect that the lattice model depends on the actual point vortex discretization of the barotropic vorticity equation. In the second subsection, we will derive a different lattice vortex gas model which is independent of the choice of species numbers and types in the particle discretization of the absolute vorticity distribution. This last model, which henceforth we will refer to as the Onsager lattice vortex gas, is preferred because the primary object of our study is the equilibrium statistics of the vorticity distribution of the barotropic vorticity equation, which should not depend on the artifact of a numerical discretization procedure.

3.1 Coarse-graining

For point vortex gas on the rotating sphere where the vortices are identical and have vorticity or charge λ, we consider the coarse-grained lattice Hamiltonian based on the division of the physical domain S^2 of area $A = 4\pi$ into M equal boxes,

$$H_0 = -\frac{1}{2}\sum_{i=1}^{M}\sum_{j\neq i}^{M} n_i n_j \lambda^2 \log(1 - \vec{x}_i^0 \cdot \vec{x}_j^0). \tag{10}$$

Here n_i denotes the number of vortices in box B_i which has area h^2, and $\vec{x}_i^0$ is the location of the center of B_i, with the constraint that the total number of particles is N, i.e.,

$$\sum_{i=1}^{M} n_i = N.$$

Thus, $\vec{x}_i^0$ in the coarse-grained Hamiltonian H_0 are no longer dependent on time but depend on a lattice on the unit sphere. The Hamiltonian (10) forms the basis of the OJM mean field theory of the barotropic vorticity equation [46], [44], [38], [31].

The nonextensive continuum limit (NCL) is given by

$$\begin{aligned} M &= 4^m M_* \to \infty \\ L &= M^{1/d} \to \infty, \\ \lambda &= 2^{-m} \lambda_* \to 0, \\ N &= 2^m N_* \\ \text{as } m &\to \infty, \end{aligned} \tag{11}$$

where the initial vorticity of the point vortices is denoted by λ_*, the initial number of lattice sites is given by M_*, L represents the diameter of the domain Ω in terms of the number of lattice sites and N_* is the initial total number of vortices. Using the method in [31], [32] it can be shown that provided we take the NCL, the error $H_1 = H - H_0$ committed in taking the coarse-grained Hamiltonian H_0 vanishes as the number of lattice sites M tends to ∞ according to $H_1 \sim M^{-1} \ln M$.

Unfortunately, if we start from the Barotropic Vorticity equation, and perform distinct vortex discretizations of the same continuous vorticity distribution (by taking two different vortex gas, for example), we end up with distinct H_0 and distinct mean field equations. It is the need to resolve this problem that motivated our formulation of another lattice vortex gas Hamiltonian which will be given in the next subsection.

3.2 Continuous spin lattice vortex gas models

When applied to the two species vortex gas on a sphere with initial vorticity λ and $-\lambda$, the above coarse-graining procedure gives the lattice Hamiltonian

$$H_0(N, M) = -\frac{1}{2} \sum_{i=1}^{M} \sum_{j \neq i}^{M} J_{ij} (n_i^+ - n_i^-)(n_j^+ - n_j^-), \tag{12}$$

where $n_i^\pm$ is the number of positive (resp. negative) vortices in box B_i, and

$$\sum_{i=1}^{M} n_i^\pm = N^\pm \tag{13}$$

are the total numbers of positive and negative vortices, and

$$\begin{aligned} J_{ij} &= \lambda^2 \log |1 - \vec{x}_i^0 \cdot \vec{x}_j^0| \\ &= \lambda^2 \log_e \sin \eta_{ij}^0 / 2. \end{aligned} \tag{14}$$

Instead of maintaining the species-specific constraints such as (13), we introduce the new spin variables

$$m_i = n_i^+ - n_i^- \tag{15}$$

and replace (13) by the local vorticity extrema constraint

$$m_i \in [-N^-, N^+], \tag{16}$$

to obtain the lattice Hamiltonian

$$H(M) = -\frac{1}{2} \sum_{i=1}^{M} \sum_{j \neq i}^{M} J_{ij} m_i m_j, \tag{17}$$

with the total circulation constraint

$$\sum_{i=1}^{M} m_i = N^+ - N^- = N^{'}. \tag{18}$$

The total number of point vortices in the problem is given by $N = N^+ + N^-$.

The constraint (18) is a total circulation type constraint, and this important constraint has been retained in going to the new spins m_i. It takes the form

$$\sum_{i=1}^{M} m_i = 0,$$

for a neutral gas of vortices. The constraint (16) limiting the values of the spins m_i (local absolute vorticity), is related to analytical results on the blow-up of solutions in the Euler equation [22], [14]. We will replace this extrema constraint with a higher dimensional spherical constraint to derive the long range spherical model. We note the important point that the spin variables m_i take on rational values in general when there are more species of vortex particles in the problem.

At this point there are three main ways to proceed from the family of lattice Hamiltonian models in (17), (14), (16) and (18): (A) impose an additional and natural constraint, namely the enstrophy-related $\sum m_i^2$, (B) obtain a mean field theory for it, and (C) solve it numerically using Monte-Carlo (as well as other) simulation methods. With the exception of a remark about the independence of this precursor Hamiltonian, and thus, its mean field theory, from the specific vortex gas discretization of the continuous vorticity field, we will present parts (B) and (C) in future papers.

3.3 Spherical models

To obtain the Spherical Model for the above neutral vortex gas with two species of vortices, we will first replace the rational valued spins m_i by the real variable

$$x_i \in [-\frac{N}{2}, \frac{N}{2}] \cap R \tag{19}$$

so that the constraint (18) becomes

$$\sum_{i=1}^{M} x_i = 0 \tag{20}$$

and the Hamiltonian takes the form

$$H(M) = -\frac{1}{2} \sum_{i=1}^{M} \sum_{j \neq i}^{M} J_{ij} x_i x_j. \tag{21}$$

As N, $M \to \infty$ in the NCL, the configurational partition function Z' of (17) converges to the configurational partition function Z of (21) by the standard properties of Riemann integration. Thus, in taking the step (19), and the domain of integration D of the configurational partition function for (21) to be

$$D = \left\{ x_i \in [-\frac{N}{2}, \frac{N}{2}], \text{ and } \sum_{i=1}^{M} x_i = 0 \right\},$$

we have not introduced any further approximation in the NCL. We have converted the problem to one with continuous spin variables x_i.

Next we let

$$D' = \left\{ x_i \in (-\infty, \infty), \ \sum_{i=1}^{M} x_i = 0, \text{ and } \sum_{i=1}^{M} x_i^2 = K > 0 \right\}. \tag{22}$$

By going from the domain D to the domain D', the bounds on the local vorticity, i.e, $x_i \in [-\frac{N}{2}, \frac{N}{2}]$ are replaced by the spherical constraint $\sum_{i=1}^{M} x_i^2 = K$, which now serves to constrain the values of x_i. From a geometric viewpoint, the old domain D is a hypercube while the new domain D' is a spherical surface in phase-space. Some such constraint on the values of the spin x_i is necessary because without them, we obtain the so-called Gaussian Model which is also exactly solvable but has a major defect that makes it unphysical [55]. This yields the Spherical Model we desired. We will show below that the spherical constraint is in fact directly related to the enstrophy constraint, thus making our long range version of Kac's Spherical model a natural model for the equilibrium statistical mechanics of 2-D ideal flows.

To summarize, we collect here the equations of this long range Spherical model for a point vortex discretization of the barotropic vorticity model:

$$
\begin{aligned}
H_S(M) &= -\frac{1}{2}\sum_{i=1}^{M}\sum_{j\neq i}^{M} J_{ij}x_i x_j \\
J_{ij}(M) &= \frac{1}{N^2}\ln(1-\vec{r}_i\cdot\vec{r}_j) \\
D' &= \left\{x_i \in (-\infty,\infty),\ \sum_{i=1}^{M} x_i = 0,\ \text{and}\ \sum_{i=1}^{M} x_i^2 = M > 0\right\}
\end{aligned} \tag{23}
$$

where D' is its domain in phase-space. We will prove next that if we had started with a different vortex gas discretization, for instance, one consisting of three species of vortices, we would have derived the same equations as in (23).

3.4 Nondegenerate Hamiltonians for the lattice vortex gas

Using the above procedure, we will construct a lattice Hamiltonian $H(M)$ for a three species point vortex discretization and compare it with the Hamiltonian of the above two species discretization of the same continuous vorticity field. We will show that the two Hamiltonians are one and the same. Their corresponding Spherical Models are therefore also the same.

We will be more careful with our notation than before so that the NCL limiting procedure in (11) is made explicit. Let the three species of vortices have initial strengths $\lambda_* = 1, \frac{\lambda_*}{2}, -\lambda_*$, in the respective initial proportions N_{1*}, $N_{1/2*}$ and N_{-1*} such that the total number of point vortices is given by

$$N_* = N_{1*} + N_{1/2*} + N_{-1*}.$$

There is a second condition on these numbers for they must satisfy the same total circulation as in the case of a two species discretization, i.e,

$$\lambda_* N_{1*} + \frac{\lambda_*}{2}N_{1/2*} - \lambda_* N_{-1*} = \lambda_*(N_*^+ - N_*^-) = \lambda_* N_*' = 0. \tag{24}$$

For the coarse-graining, we will take the occupation numbers to be specified initially by

$$n_{i*}^{(1)},\ n_{i*}^{(1/2)},\ n_{i*}^{(-1)},$$

so that

$$\begin{aligned}\sum n_{i*}^{(1)} &= N_{1*}\\ \sum n_{i*}^{(1/2)} &= N_{1/2*}\\ \sum n_{i*}^{(-1)} &= N_{-1*}.\end{aligned}$$

We obtain a different coarse-grained lattice Hamiltonian than in the two-species case.

But if we define spin variables

$$m_i = n_i^{(1)} + \frac{1}{2}\, n_i^{(1/2)} - \ n_i^{(-1)} \in Q, \tag{25}$$

we will obtain the lattice Hamiltonian

$$\begin{aligned} H(N,M) &= -\frac{1}{2}\sum_{i=1}^{M}\sum_{j\neq i}^{M} J_{ij} m_i m_j, \\ J_{ij} &= \frac{1}{N^2}\ln|1-\vec{r}_i\cdot\vec{r}_j| \end{aligned} \tag{26}$$

which is formally the same as that in (17). We note here the important point that the scaled spins m_i are just the local (lattice) values of the absolute vorticity $\omega_a(\mathbf{x},t)$ in the barotropic vorticity equation.

The zero total circulation constraint

$$\sum_{i=1}^{M} m_i = N_1 + \frac{1}{2}N_{1/2} - N_{-1} = 0 \tag{27}$$

and local vorticity extrema constraints

$$m_i \in [-N_{-1},\ N_1 + \frac{1}{2}N_{1/2}], \tag{28}$$

are preserved during the NCL procedure as $m \to \infty$. By (24), $N_{-1} = N_1 + \frac{1}{2}N_{1/2}$; thus the local vorticity extrema constraint (16) in the two species theory has the same form as that in the above three species theory, namely (28). Comparing (25), (26), (27), and (28) with the two-species Hamiltonian

(16) - (18), we conclude that both the two and three species neutral vortex gases have the same lattice Hamiltonians. We summarize in the following theorem the above discussion which can be easily made rigorous by going to arbitrary numbers of species. However we omit the details of this proof here.

Theorem 1 *The lattice vortex gas Hamiltonians (25), (26), (27), and (28) obtained from different point vortex discretizations of a given continuous absolute vorticity distribution of the barotropic vorticity equation are exactly the same.*

3.5 The enstrophy

We will now prove a key result which states that the enstrophy constraint for 2-D turbulence on the plane and the surface of a sphere is mathematically equivalent to Kac's spherical constraint $\sum_{i=1}^{M} m_i^2 = M$. The lattice form of the enstrophy constraint is given by

$$\lambda_*^2 \sum_{i=1}^{M_*} m_i^2 = K$$

in terms of the initial values of relevant quantities. Moreover, we are free to choose the initial values λ_* and M_* so that

$$\sum_{i=1}^{M_*} m_i^2 = M_* = \frac{K}{\lambda_*^2}. \tag{29}$$

Since the initial enstrophy K must be preserved as we refine the coarse-graining procedure under the NCL (where the unstarred quantities scale as in (11)), we have

$$\lambda^2 \sum_{i=1}^{M} m_i^2 = 4^{-m} \lambda_*^2 \sum_{i=1}^{4^m M_*} m_i^2 = K.$$

Then it follows immediately that

$$4^{-m} \sum_{i=1}^{M} m_i^2 = M_*$$

which in view of (11) implies

$$\sum_{i=1}^{M} m_i^2 = M \tag{30}$$

as $m \to \infty$ in the NCL. This is a key result since it clearly provides the justification for using the Spherical model to study the above Energy-Enstrophy model and the equivalent three constraints lattice model.

4 Solution of the Spherical models

The Spherical Model was introduced by Berlin and Kac [2] to provide a exactly solvable lattice Hamiltonian which is more transparent than the Ising Model in order to better understand the phase transition properties of the latter and partly to circumvent the low temperature catastrophe difficulty of the Gaussian model. In the spherical model, the Gaussian integral is replaced by a configurational integral over a higher dimensional sphere in configuration space [2] (cf. Stanley's book [55] for a nice historical note on the origins and exact solution of the spherical model). To date, the Spherical Model has been applied mainly to phase transitions in magnetic materials and also to lattice gases [56]. When the interaction in the Spherical Model is of nearest-neighbor type, it is known that a phase transition occurs only if $d \geq 3$ [56]. But for some interactions of infinite range, the Spherical Model can have a phase transition in the standard thermodynamic limit, even for $d \leq 2$ [56]. In the Onsager vortex gas and the energy-enstrophy problems on a sphere, we will show that the associated long range spherical model has a threshold temperature at $T_* = \infty$, between positive and negative temperature regimes, and also a more traditional critical temperature $0 < T_c < \infty$ where the saddle point sticks [2].

4.1 Exact solution

The partition function of the spherical model is canonical in the energy H_S and the total circulation $\sum_{i=1}^{M} x_i$, and microcanonical in the enstrophy $\sum_{j=1}^{M} x_j^2$, that is,

$$Z_S = \int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_M \exp\left[\beta \sum_{i \neq j}^{M} x_i J_{ij} x_j + \gamma \sum_{i=1}^{M} x_i\right] \delta\left(\sum_{j=1}^{M} x_j^2 - M\right) \quad (31)$$

$$= \int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_M \exp\left[\beta \sum_{i \neq j}^{M} x_i J_{ij} x_j + \gamma \sum_{i=1}^{M} x_i\right] \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} dp \exp\left[p\left(M - \sum_{j=1}^{M} x_j^2\right)\right]. \quad (32)$$

We note that the Lagrange multiplier γ conjugate to the total circulation constraint $\sum_{i=1}^{M} x_i = N'$ plays the role of the external field B in some magnetic formulations of the Spherical model. In the zero total circulation case, the above partition function becomes

$$Z_S = \int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_M \exp\left[\beta \sum_{i \neq j}^{M} x_i J_{ij} x_j\right] \delta\left(\sum_{j=1}^{M} x_j^2 - M\right). \tag{33}$$

Alternatively, we could have used the mean-spherical model formulation in which the enstrophy constraint enters into a grand-canonical partition function Z_G in a canonical way, that is, in the factor

$$\exp\left(-p \sum_{j=1}^{M} x_j^2\right).$$

Thus, for Z_G and also Z_S to be finite, it is necessary that

$$p > 0. \tag{34}$$

Since there are several sources for the exact expression of Z_S, we will only dicuss the main points below. Let $(Y_n^m(\varphi, \theta))^*$ be the complex conjugates of the surface spherical harmonics in (7) which provide a complete orthonormal system of eigenfunctions for Laplace equations on the sphere. Using the discrete Fourier transforms $a(\vec{q})$ and $z_{\vec{q}}$ of J_{jk} and $x_{\vec{j}}$

$$z_{\vec{q}} = \sum_{\vec{j} \in lattice} (Y_n^m(\varphi(\vec{j}), \theta(\vec{j})))^* x_{\vec{j}} \tag{35}$$

$$\begin{aligned} a(\vec{q}) &= \sum_{0 \neq \vec{r} \in lattice} \lambda^2 \log \sin \frac{\eta(\vec{r})}{2} (Y_n^m(\varphi(\vec{r}), \theta(\vec{r})))^* \\ &= -\lambda^2 \left(\frac{4\pi^2 n(n+1)}{L^2}\right)^{-1} < 0, \end{aligned} \tag{36}$$

where $\vec{q} = (m, n)$ (with $m = -n, ..., -1, 0, 1, ..., n$, and $n = 0, 1, , , , L-1$) are vectors in a lattice that is dual to the physical *lattice* containing the unit vectors on the unit sphere $\vec{r} \in S^2$, $\eta(\vec{r})$ is the angular separation between unit vector $\vec{r}$ and the north pole on the unit sphere, $\varphi(\vec{r}), \theta(\vec{r})$ are respectively the longitude and co-latitude at the lattice site $\vec{r}$ and $L = M^{1/2}$, we have

$$J_{jk} = \frac{1}{M} \sum_{\vec{q} \neq 0} a(\vec{q}) Y_n^m(\varphi_j, \theta_j) Y_n^{m*}(\varphi_k, \theta_k) \tag{37}$$

and a diagonalization of the interaction term

$$\beta \sum_{i \neq j}^{M} x_i J_{ij} x_j = \frac{\beta}{M} \sum_{\vec{q}} a(\vec{q}) |z_{\vec{q}}|^2.$$

The Fourier coefficient $a(\vec{0})$ for this problem is given by

$$a(\vec{0}) = \sum_{0 \neq \vec{r} \in lattice} \lambda^2 \ln \sin \left(\frac{\eta(\vec{r})}{2} \right). \tag{38}$$

The exact solution of the Spherical Model is then given by a steepest descent analysis of the following expression for the partition function:

$$Z_S = \pi^{M/2} \int dp \exp \left[M \left(p - \frac{1}{2M} \sum_{\vec{q}} \ln(p - \beta a(\vec{q})) \right) \right] \tag{39}$$

where p is a variable used in the Laplace integral representation of the spherical constraint. The partition function (39) is well-defined provided [56]

$$p - \beta \max_{\vec{q}} a(\vec{q}) > 0. \tag{40}$$

In view of the facts that p is always positive by (34), and $\max_{\vec{q}} a(\vec{q}) < 0$ by (36), we deduce that this condition (40) is always satisfied for $\beta > 0$. For $\beta \leq 0$, it is in principle possible for $p - \beta \max_{\vec{q}} a(\vec{q}) \leq 0$ for all finite values of p. This is a key point for the following analysis of the phase transitions in the energy-enstrophy model of the barotropic vorticity equation.

The value of $p = p(\beta)$ in the partition function Z_S in (39) is determined by the saddle point equation

$$1 = \frac{1}{2M} \sum_{\vec{q}} \frac{1}{(p - \beta a(\vec{q}))}. \tag{41}$$

Equivalently, p is the Lagrange multiplier conjugate to the enstrophy constraint in a grand-canonical formulation called the mean-spherical model [56]. In the latter formulation, one avoids the steepest descent analysis.

4.2 Scaling of temperature

We will show that the proper scaling of the inverse temperature β in order for the above partition function to be well-defined in the NCL is given by

$$\tilde{\beta} = \frac{\beta}{M}$$

where M is the number of lattice sites. Starting with the saddle point criterion in the NCL which takes the form

$$
\begin{aligned}
1 &= \frac{1}{8\pi}\int_0^{2\pi} d\phi_2 \int_0^{\phi_2} d\phi_1 \frac{1}{p-\beta a(\vec{\phi})} \\
&= \frac{1}{8\pi}\int_0^{2\pi} d\phi_2 \int_0^{\phi_2} d\phi_1 \frac{1}{p+\frac{\beta}{M}\frac{\lambda_*^2 M_*}{N_*^2}\left(\frac{4\pi^2 n(n+1)}{L^2}\right)^{-1}}
\end{aligned}
$$

after using the scaling in (11), we see that the inverse temperature appears in the form of $\frac{\beta}{M}$. This implies that we scale $\beta = \tilde{\beta} M$.

4.3 Free energy

From (39), the free energy F of the above long range Spherical Model in the zero total circulation (neutral vortex gas) case is given in the NCL by

$$
-\tilde{\beta}F = -\frac{(2\pi)^{-2}}{2}\int_0^{2\pi} d\phi_2 \int_0^{\phi_2} d\phi_1 \ln\left(p+\tilde{\beta}\phi_2^{-2}\right) - \frac{1}{2}\ln\tilde{\beta} + \frac{1}{2}\ln\pi \tag{42}
$$

where $\tilde{\beta} = \frac{\beta}{M}$ is the properly scaled inverse temperature for the NCL (11) and $\vec{\phi} = (\phi_1, \phi_2)$ and the angles ϕ_j are defined here to be

$$
\begin{aligned}
\phi_1 &= \frac{2\pi q_1}{L}, \\
\phi_2 &= \frac{2\pi q_2}{L}
\end{aligned}
\tag{43}
$$

with $q_1 = m \in \{-n, ..., -1, 0, 1, \ldots, n\}$, $q_2 = n \in \{0, 1, \ldots, L-1\}$, and $L = M^{1/2}$ is the square root of the number of lattice sites. For $\tilde{\beta} > 0$, we define $\tilde{\zeta} = \frac{p}{\tilde{\beta}} > 0$, and the free energy (42) becomes

$$
\begin{aligned}
-\tilde{\beta}F &= -\frac{1}{2}\ln\tilde{\beta} + \frac{1}{2}\ln\pi \\
&\quad -\frac{(2\pi)^{-2}}{2}\int_0^{2\pi} d\phi_2 \int_0^{\phi_2} d\phi_1 \ln\left(\tilde{\zeta}+\phi_2^{-2}\right),
\end{aligned}
$$

to make contact with usual nomenclature.

For (42) to be well-defined, it follows from (40) that we must have

$$
p > -\tilde{\beta} \lim_{L\to\infty} \max_{n>0} \left(\frac{4\pi^2 n(n+1)}{L^2}\right)^{-1}. \tag{44}
$$

Moreover the saddle point equation must be satisfied:

$$1 = \bar{R}(p, \tilde{\beta}) \equiv \frac{(2\pi)^{-2}}{2} \int_0^{2\pi} d\phi_2 \int_0^{\phi_2} \frac{d\phi_1}{p + \tilde{\beta}\phi_2^{-2}}. \tag{45}$$

Since (40) is always satisfied for $\tilde{\beta} > 0$, the free energy (42) is valid for positive temperatures provided that the saddle point equation is satisfied.

5 Phase transitions

There are two conditions for the free energy (42) to be well-defined, namely (i) the inequality (40) and (ii) the saddle point equation (45). We will see that both enter into the analysis of the phase transitions of this problem. It is convenient to separate the calculations into two parts, for positive and negative temperatures respectively. We will see that for positive temperatures, the saddle point equation is the determining factor since (40) is trivially true. In this case, we will set $p = 0$ to solve (45) for the most positive allowed value of $\tilde{\beta}$, that is $\tilde{\beta}_c$. On the other hand, we will see that one can use condition (i) for the analysis of non-positive temperatures. Here we will set $p = \infty$ to find the most negative allowed value of $\tilde{\beta}$, that is $\tilde{\beta}_* \leq 0$. Collecting the results from these two subsections, we have the result:

Theorem 2 *The partition function Z_S of the zero total circulation spherical model (23) for the energy-enstrophy theory of the barotropic vorticity equation, is analytic in the inverse temperature β in the range*

$$\frac{N_*^2\pi^2}{2\lambda_*^2 M_*} = \tilde{\beta}_c > \tilde{\beta} > \tilde{\beta}_* = 0.$$

Outside of this range we will use another method based on first isolating the largest terms in the sums in (39) and (41) and then taking the $M \to \infty$ limit.

5.1 Positive temperatures

In this subsection, we will compute the critical temperature $\tilde{T}_c$ of the long range Spherical model (23) in the nonextensive continuum limit (NCL) given by (11). We will show below that in the NCL the long range Spherical model (23) has $\tilde{\beta}_c = \frac{N_*^2\pi^2}{2\lambda_*^2 M_*}$ in the properly scaled inverse temperature $\tilde{\beta} = \frac{\beta}{M}$.

We will now compute the upper bound $\tilde{\beta}_c$ by setting $p = 0$ in the saddle point equation (45) to obtain

$$\begin{aligned}
\tilde{\beta}_c &= \frac{N_*^2}{8\pi^2\lambda_*^2 M_*}\int_0^{2\pi} d\phi_2 \int_0^{\phi_2} \frac{d\phi_1}{\phi_2^{-2}} \\
&= \frac{N_*^2}{8\pi^2\lambda_*^2 M_*}\int_0^{2\pi} d\phi_2 \frac{1}{\phi_2^{-2}} \int_0^{\phi_2} d\phi_1 \\
&= \frac{N_*^2}{8\pi^2\lambda_*^2 M_*}\int_0^{2\pi} \phi_2^3 \, d\phi_2 \\
&= \frac{N_*^2\pi^2}{2\lambda_*^2 M_*}
\end{aligned}$$

We have thus proved the following result.

Theorem 3 *In the NCL (11), the energy-enstrophy model (23) of the barotropic vorticity equation has a positive critical inverse temperature*

$$0 < \tilde{\beta}_c = \frac{N_*^2\pi^2}{2\lambda_*^2 M_*}, \tag{46}$$

where $\tilde{\beta} = \frac{\beta}{M}$ is the properly scaled inverse temperature.

We will argue in the section on physical interpretations that the above finite positive value of $\tilde{\beta}_c = \frac{N_*^2\pi^2}{2\lambda_*^2 M_*}$ is an artificial upper bound on the inverse temperature, and by suitable choices of the initial values N_* and λ_*, one can make $\tilde{\beta}_c$ as large as we wish. Nonetheless, there remains the problem of the low temperature behaviour of the free energy for $0 < \tilde{T} < \tilde{T}_c$ because no matter how small we make $\tilde{T}_c$, it is still strictly positive. We will come back to this issue later.

5.2 Negative temperatures

We will show below that in the NCL, the inequality (40) can be solved for $p > 0$ only for $\tilde{\beta} \geq 0$. From the fact that for any $\tilde{\beta} < 0$

$$-\tilde{\beta} \lim_{L\to\infty} \max_{n>0} \left(\frac{4\pi^2 n(n+1)}{L^2}\right)^{-1} = +\infty$$

in inequality (44), it follows that it is impossible to satisfy (44) with finite values of p. We can view this result from the point of view of setting $p = \infty$

in the inequality (44) to get $\tilde{\beta}_* = 0$, which is identified with $\tilde{T}_* = \infty$ and $\tilde{T}_* = -\infty$.

The boundary value $\tilde{T}_* = \infty$ agrees with several 2-dimensional computational efforts [41], [8], [51], [6] indicating a change to coherent structures when the temperature goes from $\tilde{T} < \infty$ through $\tilde{T}_* = \infty$ and $\tilde{T}_* = -\infty$ to hotter $-\infty < \tilde{T} < 0$. The two-point correlations for different temperature regimes to be calculated next, will serve as a diagnostic for this qualitative change.

6 Correlations

In this section we calculate the two-point spin-spin correlations $\langle x_i x_j \rangle$ which are related to the two-point vorticity correlations in the equilibrium vorticity distributions predicted by the above results. These vorticity correlations are also related to the structure functions that are ubiquitous in the study of turbulence [13], [14], [15], [53], [47]

. We will calculate these correlations for three distinct temperature regimes: (i) $\tilde{T} > \tilde{T}_c$, (ii) $0 < \tilde{T} < \tilde{T}_c$ and (iii) $\tilde{T} < 0$.

6.1 High positive temperatures $\tilde{T} > \tilde{T}_c$

In the limit $M \to \infty$, the correlations $\langle x_i x_j \rangle = \frac{1}{\beta} \frac{\partial \ln Z_S}{\partial J_{ij}}$ are given by

$$\langle x_i x_j \rangle = \frac{(2\pi)^{-2}}{2\tilde{\beta}} \int_0^{2\pi} d\phi_2 \int_0^{\phi_2} d\phi_1 \frac{e^{im(\varphi_j - \varphi_i)} P_n^m(\cos\theta_i) P_n^m(\cos\theta_j)}{\tilde{\zeta} + \phi_2^{-2}}. \quad (47)$$

The saddle point equation holds for $\tilde{\beta} < \tilde{\beta}_c$, and the expression (47) makes sense for the saddle point $\tilde{\zeta}(\tilde{\beta}) > \tilde{\zeta}_c = 0$. It follows from (47) that the correlations $\langle x_i x_j \rangle \simeq 0$ at fixed lattice sites i and j. For finite values of M, and $p(\tilde{\beta})$ computed from (45),

$$\langle x_i x_j \rangle \sim \frac{1}{2M} \sum_{n=1}^{L-1} \left(p + \tilde{\beta} \frac{L^2}{4\pi^2 n(n+1)} \right)^{-1} \sum_{m=-n}^{n} e^{im(\varphi_j - \varphi_i)} P_n^m(\cos\theta_i) P_n^m(\cos\theta_j), \quad (48)$$

which is a key expression for the numerical computations of correlations in this problem. We hope to return to the idea of using the finite M formulae to compute various thermodynamic quantities approximately in another paper.

6.2 Low positive temperatures $0 < \tilde{T} < \tilde{T}_c$

For the low temperature properties of the free energy (42) and the two-point spin-spin correlations $\langle x_i x_j \rangle$, we will follow Thompson [52] in extracting the dominant terms in the sum in the expression (41) for the saddle point $\tilde{\zeta}(\tilde{\beta}) \equiv \frac{p}{\tilde{\beta}}$,

$$\tilde{\beta} = \frac{1}{2M} \frac{N_*^2}{\lambda_*^2 M_*} \sum_{2\pi\vec{q}/L} \left[\tilde{\zeta} + \left(\frac{4\pi^2 n(n+1)}{L^2} \right)^{-1} \right]^{-1} \tag{49}$$

and then take the limit $M \to \infty$.

Since $\tilde{\beta}_c = \frac{N_*^2 \pi^2}{2\lambda_*^2 M_*}$ we have $\tilde{\zeta}_c = 0$, and equation (49) becomes for large M,

$$\tilde{\beta} = \frac{1}{2M} \frac{N_*^2}{\lambda_*^2 M_* (\tilde{\zeta} - 0)} + R(0) \tag{50}$$

where

$$R(\tilde{\zeta}) = \frac{(2\pi)^{-2}}{2} \int_0^{2\pi} d\phi_2 \int_0^{\phi_2} \frac{d\phi_1}{\tilde{\zeta} + \phi_2^{-2}}.$$

For fixed $\tilde{\zeta}$ and $M \to \infty$, equation (50) reduces to (45) which determines $\tilde{\zeta}(\tilde{\beta})$ when $\tilde{\beta} < \tilde{\beta}_c$. For $\tilde{\beta} > \tilde{\beta}_c$, we set

$$\tilde{\zeta} = \frac{2\lambda_*^2 M_*}{N_*^2} \frac{1}{bM},$$

before taking the $M \to \infty$ limit and write (50) in the form

$$\tilde{\beta} = b + \tilde{\beta}_c.$$

Thus the quantity b is now given by

$$b = \tilde{\beta} - \tilde{\beta}_c > 0$$

when $\tilde{\beta} > \tilde{\beta}_c$, which shows in fact that the saddle point $\tilde{\zeta}$ sticks at the value $\tilde{\zeta}_c = 0$.

When M is finite, the spin-spin correlations for $0 < \tilde{T} < \tilde{T}_c$ are given by (48). By carrying out the above procedure on (48), that is, separating out the dominant terms and then taking the limit $M \to \infty$, we obtain the expression

$$\langle x_i x_j \rangle = \left(1 - \frac{\tilde{T}}{\tilde{T}_c} \right) \int_0^{2\pi} d\phi_1^0 e^{im_0(\varphi_j - \varphi_i)} P_{n_0}^{m_0}(\cos\theta_i) P_{n_0}^{m_0}(\cos\theta_j)$$
$$+ \frac{(2\pi)^{-2}}{2\tilde{\beta}} \int_0^{2\pi} d\phi_2 \int_0^{\phi_2} d\phi_1 \frac{e^{im(\varphi_j - \varphi_i)} P_n^m(\cos\theta_i) P_n^m(\cos\theta_j)}{\tilde{\zeta}_c + \phi_2^{-2}}$$

$$= \left(1 - \frac{\tilde{T}}{\tilde{T}_c}\right) \int_0^{2\pi} d\phi_1^0 e^{im_0(\varphi_j - \varphi_i)} P_{n_0}^{m_0}(\cos\theta_i) P_{n_0}^{m_0}(\cos\theta_j) \quad (51)$$

$$+ \frac{1}{8\pi^2 \tilde{\beta}} \int_0^{2\pi} d\phi_2 \frac{1}{\phi_2^{-2}} \int_0^{\phi_2} d\phi_1 e^{im(\varphi_j - \varphi_i)} P_n^m(\cos\theta_i) P_n^m(\cos\theta_j)$$

where $\vec{\phi}^0 = 2\pi(m_0, n_0)/L$ corresponds to the dominant terms in the sum (48). These dominant terms are determined by setting $\tilde{\zeta}_c = 0$ in the denominator of the sum in (48), and the values (m_0, n_0) that maximizes $-\left(\frac{4\pi^2 n(n+1)}{L^2}\right)^{-1}$ are $n_0 = L - 1$ with any m_0 in the range $0 \leq |m_0| \leq n_0$. This accounts for the fact that the leading term in (51) is an integral over the longitudinal (or zonal) wave-number $\phi_1 = 2\pi m_0/L$.

It is clear from (51) that the leading term gives the main contribution to the correlations $\langle x_i x_j \rangle$, that is,

$$\langle x_i x_j \rangle \sim \left(1 - \frac{\tilde{T}}{\tilde{T}_c}\right) \int_0^{2\pi} d\phi_1^0 e^{im_0(\varphi_j - \varphi_i)} P_{n_0}^{m_0}(\cos\theta_i) P_{n_0}^{m_0}(\cos\theta_j). \quad (52)$$

This expression implies that as $\tilde{T} \nearrow \tilde{T}_c$, the correlation $\langle x_i x_j \rangle$ for fixed lattice sites $\vec{r}_i$ and $\vec{r}_j$ on the surface of the unit sphere, tends to zero. We also deduce that the spatial dependence of $\langle x_i x_j \rangle$ on $\vec{r}_i$ and $\vec{r}_j$ is the same at all low temperatures $\tilde{T} < \tilde{T}_c$ since this spatial dependence is determined by the integral $\int_0^{2\pi} d\phi_1^0 e^{im_0(\varphi_j - \varphi_i)} P_{n_0}^{m_0}(\cos\theta_i) P_{n_0}^{m_0}(\cos\theta_j)$ of the surface spherical harmonics over the longitudinal wavenumber ϕ_1^0. These harmonics correspond to the largest latitudinal wavenumber $\phi_2 = 2\pi$, which means that the associated correlations $\langle x_i x_j \rangle$ have the shortest allowed wavelength in the latitudinal direction. On the other hand, they range from being axisymmetric about the axis of rotation for $m_0 = 0$, to having the shortest allowed wavelength in the longitudinal (or zonal) direction. We will have more to say about the physical significance of these results in the next section.

From a numerical point of view the expression

$$\langle x_i x_j \rangle \sim \left(1 - \frac{\tilde{T}}{\tilde{T}_c}\right) \frac{1}{L} \sum_{m=-L+1}^{L-1} e^{im(\varphi_j - \varphi_i)} P_{L-1}^m(\cos\theta_i) P_{L-1}^m(\cos\theta_j)$$

provides an important way to compute the correlations $\langle x_i x_j \rangle$ in the low positive temperature regime.

6.3 Negative temperatures $\tilde{T} < 0$

In this subsection we will discuss the mathematical and physical consequences of negative temperature which underlies much of the remarkable properties of the vortex gas models for 2-D ideal fluids. From the section on phase transitions, we calculated a boundary temperature $\tilde{T}_* = \pm\infty$ by showing that the expression (44) cannot be satisfied for any $\tilde{\beta} < 0$. The correlations in this regime cannot be calculated via (47) because the saddle point equation (45) fails to be valid when for fixed p the expression $p + \tilde{\beta}\phi_2^{-2} < 0$ over the small wavenumber part of the spectrum.

For finite M, the correlations $\langle x_i x_j \rangle$ are given by equation (48), and since the denominators $p+\tilde{\beta}\frac{L^2}{4\pi^2 n(n+1)}$ are not all positive for fixed $p(\tilde{\beta})$ when $\tilde{\beta} < 0$, we will extract the dominant terms before taking the limit $M \to \infty$ as in [52]. After extracting the dominant term given by $n = 1$, we get from (48),

$$\langle x_i x_j \rangle \sim \frac{1}{2M}\left(p + \tilde{\beta}\frac{L^2}{8\pi^2}\right)^{-1} P_1^0(\cos\theta_i)P_1^0(\cos\theta_j) \tag{53}$$

$$+\frac{1}{2M}\sum_{n=2}^{L-1} \left(p(M) + \tilde{\beta}_*(M)\frac{L^2}{4\pi^2 n(n+1)}\right)^{-1} \sum_{m=-n}^{n} e^{im(\varphi_j - \varphi_i)} P_n^m(\cos\theta_i)P_n^m(\cos\theta_j).$$

This procedure yields in the limit of large M

$$\begin{aligned}\langle x_i x_j \rangle &\sim \frac{\tilde{\beta} - \tilde{\beta}_*}{\tilde{\beta}} P_1^0(\cos\theta_i)P_1^0(\cos\theta_j) \\ &= P_1^0(\cos\theta_i)P_1^0(\cos\theta_j),\end{aligned} \tag{54}$$

and the rest of the sum in (54) vanishes since the denominator behaves like

$$p(M) + \tilde{\beta}_*(M)\frac{L^2}{4\pi^2 n(n+1)} \to \infty.$$

The physical interpretations for the expression (54) will discussed in the next section.

7 Physical interpretations

In this section, we will give physical interpretations of the mathematically exact result in this paper. This interpretation must begin in some sense with the justification for taking a equilibrium statistical mechanical approach in a field of study where dynamics play an important role. For clear expositions of such justification we need go no further than the historic work of Batchelor (Chap 6 of [1]). We also refer the reader to the discussions by Chorin [8] on near-equilibrium theories of turbulence. We summarize Batchelor's comments here which are based on the work of Kolmogorov [26] and Onsager [46]. The assumption of equilibrium statistical mechanics is justified when the time scale of the over-all decay of energy is much longer than the characteristic time (turnover time) of the energy-containing eddies. It is argued that the characteristic time should decrease with the size of eddies; thence for a range of sufficiently large wavenumbers k, the assumption of equilibrium is justified because the time scale of over-all decay of energy is independent of wavenumber k.

A second issue that arise in the use of equilibrium statistical mechanics in the study of macroscopic fluid motions, is the physical interpretation of temperature. We stress here that the temperature $\tilde{T}$ in this paper is firstly, a mathematical construct in the sense that $\tilde{\beta} = 1/\tilde{T}$ is the Lagrange multiplier conjugate to the energy in a Gibbs ensemble, and secondly, a physical measure of the macroscopic motions of the fluid which is treated as a continuum. This temperature is not related at all to the *temperature* of the molecular motions of the fluid. The shortest time scale of the macroscopic flow of fluids is many orders of magnitude greater than the molecular relaxation time.

Another analogy that is worth pointing out is that the logarithmic vortex gas interaction J_{ij} is antiferromagnetic instead of ferromagnetic since J_{ij} is predominantly negative. This agrees with some analytical and numerical results[8], [46], [33] indicating that for $\tilde{\beta} > 0$, the lattice vortex gas model has equilibrium states which are disordered, while for $\tilde{\beta} < 0$, its equilibrium states consist of macroscopic coherent structures.

Moreover, from equation (29), we have a relation $K = M_* \lambda_*^2$ between the fixed value of the enstrophy K and the initial values M_* and λ_*. Substituting for $M_* \lambda_*^2$ in $\tilde{\beta}_c = \frac{N_*^2 \pi^2}{2\lambda_*^2 M_*}$, we obtain the expression

$$\tilde{\beta}_c = \frac{N_*^2 \pi^2}{2K},$$

which links the value of the enstrophy K to the critical temperature $\tilde{\beta}_c$. It is clear that the rotation rate Λ of the sphere enters into this expression through the enstrophy $K(\Lambda, \omega_r)$. For example, when the relative vorticity ω_r is zero, K is completely determined by Λ.

From this result, it follows that the critical temperature $\tilde{T}_c$ can be made equal to any positive value by a suitable choice of the initial values for the total number of point vorices N_*. To make $\tilde{T}_c$ as small as we wish, choose N_* large and λ_* small in proportion so that $N_*\lambda_*$ remains constant. It is clear that one can always do this; in fact one must choose N_* and λ_* so that $N_*\lambda_*$ remain fixed because if we increase the initial vortex strength scale λ_*, we must decrease the initial total number of vortices N_* used in the statistical mechanics formulation (or the initialization step in a numerical computation) in inverse proportion in order to model the same continuum vorticity distribution.

The physical significance of the results derived in the section on correlations will be discussed next. First, the correlations for low positive temperatures $0 < \tilde{T} < \tilde{T}_c$ in expression (52) and the discussion that followed, can be interpreted as indicative of a banded zonal structure with very short wavelength variations in the latitude, and more complicated spatial structure in the longitude. This has been seen in planetary atmospheres such as the Jovian one. The spatial variations in the longitudinal direction (along a fixed circle of latitude) is more complex because the expression (52) contains a sum (over the longitudinal wavenumber ϕ_1) of all the surface spherical harmonics $Y_{2\pi}^m$ that are associated with the shortest wavelength mode ($\phi_2 = 2\pi$) in the latitudinal direction. There is the lowest mode corresponding to $m = 0$, which has no spatial dependence on the longitude φ, at one end; and the highest mode with $m = 2\pi$ which has the shortest wavelengths in the longitude. The lowest mode $m = 0$ is associated with the axisymmetric planetary component of the absolute vorticity ω_a in (2) while the higher modes correspond to the symmetry-breaking components of relative vorticity ω_r. Their dependence on the longitude φ is through the factor $e^{im\varphi}$ in the harmonics Y_{L-1}^m. The same physical interpretations hold for high positive temperatures $\infty > \tilde{T} > \tilde{T}_c$.

At $\tilde{T}_c > 0$ there is a transition to the high positive temperatures regime where the absolute vorticity correlations $< x_i x_j >$ are close to zero. This is the phase where the equilibrium vorticity configurations are dominated by random distributions of absolute vorticity of opposite signs. Since our discussion above indicates that the value of $\tilde{T}_c$ can be made arbitrarily large (or small) by changing the value of one parameter, namely N_*, it is not clear whether $\tilde{T}_c$ corresponds to a physically meaningful phase transition, despite the fact that there is obviously a qualitative change in the correlations at $\tilde{T}_c$:

from near zero above $\tilde{T}_c$ to a oscillatory pattern in the longitudinal direction with a very short wavelength banded structure in the latitudinal direction. This is followed by a transition at $\tilde{T}_* = \infty$ to axisymmetric coherence in the high energy phase.

For negative temperatures, our calculations above indicate that the spin-spin correlations are given by (54) which states that the two-point vorticity correlations have the spatial dependence of the surface harmonic with the longest wavelength in the latitude θ, that is, the Legendre polynomial

$$\begin{aligned} \langle x_i x_j \rangle &= P_1^0(\cos\theta_i) P_1^0(\cos\theta_j) \\ &= Q \cos\theta_i \cos\theta_j \end{aligned}$$

where Q is a constant. It has no spatial dependence on the longitude φ, that is, $\langle x_i x_j \rangle$ is axisymmetric about the polar axis. It is correct to call this macrostate the ground state; its vorticity distribution is proportional to the planetary voricity $\omega_p = 2\Omega\cos\theta$. Moreover, expression (54) implies that $\langle x_i x_j \rangle$ is independent of the actual value of the negative temperature $\tilde{T} < 0$. Again, this result is consistent with known numerical and analytical results for the planar problem. It remains to compare these equilibrium absolute vorticity distributions with the numerical results in Tang and Orszag 1998, Rhines 1975, and Bretherton and Haidvogel 1976 (cf. [28] for more references).

The correlations $\langle x_i x_j \rangle$ at negative temperatures indicate a lowest mode (condensed mode) which is axisymmetric about the axis of rotation of the sphere. This implies that the coherent vortex is most likely a atmospheric formation such as the winter polar vortex. It has been suggested however that Jupiter's Giant Red Spot is a long-lived coherent structure created largely by the 2-D inviscid dynamics of the Jovian atmosphere. This conjecture is not supported by our exact solutions of the Energy-Enstrophy theory of the BVE. It is possible that with the inclusion of surface topography and other pre-conditioning effects [17], a modified BVE model may have a non-axisymmetric coherent structure such as the Red Spot.

8 Concluding remarks and future work

This new formulation of 2-D turbulence within the framework of the barotropic vorticity equation is based on a lattice vortex gas Hamiltonian like the Ising model in some respects, but with continuous spins. It was shown above that by employing properly scaled spin variables m_i instead of the bare occupation numbers in our formulation, the degeneracy problem in the Onsager theory has been avoided. The lattice vortex gas Hamiltonians derived above form a family of related models, differing only in the number of conserved quantities that are explicitly incorporated. If only the energy and total circulations are used, then one obtains essentially a version of the Onsager theory but without the degeneracy problem. By going to three constraints, namely energy, total circulation and enstrophy, all of which are natural in two-dimensional ideal fluid dynamics, we showed that the resulting lattice vortex gas model is mathematically equivalent to the spherical model, provided that a nonextensive continuum limiting procedure (NCL) is used to refine the coarse-grained lattice. This version of the spherical model has long range logarithmic interactions. It differs from other applications of the spherical model, in the sense that the spherical constraint is not an approximation but rather an exactly conserved natural quantity in the theory of ideal 2-dimensional turbulence. A calculation of the critical temperature for this model in the NCL showed that $\tilde{T}_* = \infty$. But in view of the fact that this spherical model supports negative temperatures (which are associated with higher energies, and thus considered to be "hotter" than positive temperatures), $\tilde{T}_* = \infty$ is the boundary between the occurence of a large-scale coherent vortex at negative temperatures, and a random distribution of absolute vorticity at positive temperatures. The two-point spin-spin correlations computed directly from the exact solution for the free energy of this spherical model reflect this dichotomy between a macroscopic coherent structutre at negative temperatures and a rather uncorrelated distribution of absolute vorticity at positive temperatures. A more traditional critical temperature $\tilde{T}_c > 0$ was also found and shown to be dependent on the initial values N_* and λ_* of the lattice vortex gas. This allows $\tilde{T}_c$ to be adjusted upwards to any large positive value, and also downwards to any small positive value.

Through a Fourier transform [34], [27], one can show that this three constraints lattice vortex gas is equivalent to the truncated energy-enstrophy model for the barotropic vorticity equation. Hence our exact solution of the equilibrium statistics of the lattice vortex gas gives a exact solution of the so-called absolute statistical mechanics of the well-known energy-enstrophy theory for the rotating sphere (cf. [28] and the references therein) within the spherical model framework. Our solutions differ significantly from the Gaussian solutions for the 2-D Euler problem in [27] and those for the rotating sphere as reviewed in [28]; our method yields a phase transition at the critical temperature $\tilde{T}_* = \infty$ between random signed vorticity distributions at positive temperatures and coherent structures at negative ones. These absolute vorticity distributions at different energies and enstrophies were calculated exactly using the two-point correlations of the spherical model. The Gaussian solutions in [27] are not even defined at certain energies because of the low temperature catastrophe discovered by Kac [2].

It has been shown by Stanley [54] that the spherical model lattice Hamiltonian is the infinite n limit of a family of $n - vector$ Hamiltonian models or Heisenberg models [56]. We hope to use this relationship to study more complex quasi-two dimensional models of turbulence that arise in quasi-geostrophic theories of open ocean convection [21], [36], [48], [12], [16], [17], and baroclinic two-layer models in atmospheric sciences [58] in future papers. A sequel of this paper will be devoted to the derivation of the k^{-3} and $k^{-5/3}$ energy spectrum of 2-D and BVE turbulence from the exact expressions for the two-point vorticity correlations of the spherical model [3]. The Nastrom-Gage energy spectrum in atmospheric turbulence[58] will be studied by an application of the Heisenberg $2 - vector$ model.

We will also return to the formulation and possible exact solutions of a family of lattice vortex models for the 3-D Euler equations, borrowing ideas from the work of Chorin [8], [11].

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