Real Analysis  HWk #2 for semester

Due in seminar one week from assignment.

Do all circled probs.
Unfortunately, as we shall see in Section 4, it is impossible to construct a set function having all four of these properties, and it is not known whether there is a set function satisfying the first three properties. Consequently, one of these properties must be weakened, and it is most useful to retain the last three properties and to weaken the first condition so that \( mE \) need not be defined for all sets \( E \) of real numbers. We shall want \( mE \) to be defined for as many sets as possible and will find it convenient to require the family \( \mathcal{M} \) of sets for which \( m \) is defined to be a \( \sigma \)-algebra. Thus we shall say that \( m \) is a countably additive measure if it is a nonnegative extended real-valued function whose domain of definition is a \( \sigma \)-algebra \( \mathcal{M} \) of sets (of real numbers) and we have \( m(\bigcup E_n) = \sum mE_n \) for each sequence \( \langle E_n \rangle \) of disjoint sets in \( \mathcal{M} \). Our goal in the next two sections will be the construction of a countably additive measure which is translation invariant and has the property that \( mI = l(I) \) for each interval \( I \).

Problems

Let \( m \) be a countably additive measure defined for all sets in a \( \sigma \)-algebra \( \mathcal{M} \).

1. If \( A \) and \( B \) are two sets in \( \mathcal{M} \) with \( A \subseteq B \), then \( mA \leq mB \). This property is called monotonicity.

2. Let \( \langle E_n \rangle \) be any sequence of sets in \( \mathcal{M} \). Then \( m(\bigcup E_n) \leq \sum mE_n \). [Hint: Use Proposition 1.2.] This property of a measure is called countable subadditivity.

3. If there is a set \( A \) in \( \mathcal{M} \) such that \( mA < \infty \), then \( m\emptyset = 0 \).

4. Let \( nE \) be \( \infty \) for an infinite set \( E \) and be equal to the number of elements in \( E \) for a finite set. Show that \( n \) is a countably additive set function that is translation invariant and defined for all sets of real numbers. This measure is called the counting measure.

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1 If we assume the continuum hypothesis (that every noncountable set of real numbers can be put in one-to-one correspondence with the set of all real numbers), then such a measure is impossible.

2 Weakening property (i) is not the only approach; it is also possible to replace property (iii) of countable additivity by the weaker property of finite additivity: For each finite sequence \( \langle E_n \rangle \) of disjoint sets we have \( m(\bigcup E_n) = \sum mE_n \) (see Problem 10.21). Another possible alternative to property (iii) is countable subadditivity, which is satisfied by the outer measure we construct in the next section (see Problem 2).
Since $\epsilon$ was an arbitrary positive number,

$$m^*(\bigcup A_n) \leq \sum m^*A_n.$$  

3. **Corollary:** If $A$ is countable, $m^*A = 0$.

4. **Corollary:** The set $[0, 1]$ is not countable.

5. **Proposition:** Given any set $A$ and any $\epsilon > 0$, there is an open set $O$ such that $A \subset O$ and $m^*O \leq m^*A + \epsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*A = m^*G$.

Problems

5. Let $A$ be the set of rational numbers between 0 and 1, and let \{I_n\} be a finite collection of open intervals covering $A$. Then $\sum l(I_n) \geq 1$.

6. Prove Proposition 5.

7. Prove that $m^*$ is translation invariant.

8. Prove that if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.

3 Measurable Sets and Lebesgue Measure

While outer measure has the advantage that it is defined for all sets, it is not countably additive. It becomes countably additive; however, if we suitably reduce the family of sets on which it is defined. Perhaps the best way of doing this is to use the following definition due to Carathéodory:

**Definition:** A set $E$ is said to be measurable if for each set $A$ we have $m^*A = m^*(A \cap E) + m^*(A \cap \overline{E})$.

Since we always have $m^*A \leq m^*(A \cap E) + m^*(A \cap \overline{E})$, we see that $E$ is measurable if (and only if) for each $A$ we have $m^*A \geq m^*(A \cap E) + m^*(A \cap \overline{E})$. Since the definition of measurability is symmetric in $E$ and $\overline{E}$, we have $\overline{E}$ measurable whenever $E$ is. Clearly $\emptyset$ and the set $\mathbb{R}$ of all real numbers are measurable.

6. **Lemma:** If $m^*E = 0$, then $E$ is measurable.

\footnote{In the present case $m^*$ is Lebesgue outer measure, and we say that $E$ is Lebesgue measurable. More general notions of measurable set are considered in Chapters 11 and 12.}
Hwk 2 for presentation Thur Sept 13

Do circled probs.
Problems

9. Show that if \( E \) is a measurable set, then each translate \( E + y \) of \( E \) is also measurable.

10. Show that if \( E_1 \) and \( E_2 \) are measurable, then
\[
m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2.
\]

11. Show that the condition \( mE_n < \infty \) is necessary in Proposition 14 by giving a decreasing sequence \( \langle E_n \rangle \) of measurable sets with \( \emptyset = \bigcap E_n \) and \( mE_n = \infty \) for each \( n \).

12. Let \( \langle E_i \rangle \) be a sequence of disjoint measurable sets and \( A \) any set.
Then \[
m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i).
\]

13. Prove Proposition 15. [Hints:
   a. Show that for \( m^*E < \infty \), (i) \( \Rightarrow \) (vi) \( \Leftrightarrow \) (cf. Proposition 5).
   b. Use (a) to show that for arbitrary sets \( E \), (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i).
   c. Use (b) to show that (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (v) \( \Rightarrow \) (i).]

14. a. Show that the Cantor ternary set (Problem 2.37) has measure zero.
    b. Let \( F \) be a subset of \([0, 1]\) constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the \( n \)th step has length \( \alpha 3^{-n} \) with \( 0 < \alpha < 1 \). Then \( F \) is a closed set, \( \bar{F} \) dense in \([0, 1]\) and \( mF = 1 - \alpha \). Such a set \( F \) is called a generalized Cantor set.

*4 A Nonmeasurable Set

We are going to show the existence of a nonmeasurable set. If \( x \) and \( y \) are real numbers in \([0, 1]\), we define the sum modulo 1 of \( x \) and \( y \) to be \( x + y \), if \( x + y < 1 \), and to be \( x + y - 1 \) if \( x + y \geq 1 \). Let us denote the sum modulo 1 of \( x \) and \( y \) by \( x \mod y \). Then \( \mod \) is a commutative and associative operation taking pairs of numbers in \([0, 1]\) into numbers in \([0, 1]\). If we assign to each \( x \in [0, 1] \) the angle \( 2\pi x \), then addition modulo 1 corresponds to the addition of angles. If \( E \) is a subset of \([0, 1]\), we define the translate modulo 1 of \( E \) to be the set \( E \mod y = \{ z : z = x \mod y \text{ for some } x \in E \} \). If we consider addition modulo 1 as addition of angles, translation modulo 1 by \( y \) corresponds to rotation through an angle of \( 2\pi y \). The following lemma shows that Lebesgue measure is invariant under translation modulo 1.

16. **Lemma**: Let \( E \subseteq [0, 1] \) be a measurable set. Then for each \( y \in [0, 1] \) the set \( E \mod y \) is measurable and \( m(E \mod y) = mE \).
except on a set of measure less than \( \epsilon \); i.e., \( m\{x: |f(x) - g(x)| \geq \epsilon \} < \epsilon \) and \( m\{x: |f(x) - h(x)| \geq \epsilon \} < \epsilon \). If in addition \( m \leq f \leq M \), then we may choose the functions \( g \) and \( h \) so that \( m \leq g \leq M \) and \( m \leq h \leq M \).

If \( A \) is any set, we define the \textbf{characteristic function} \( \chi_A \) of the set \( A \) to be the function given by

\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A.
\end{cases}
\]

The function \( \chi_A \) is measurable if and only if \( A \) is measurable. Thus the existence of a nonmeasurable set implies the existence of a nonmeasurable function.

A real-valued function \( \varphi \) is called \textbf{simple} if it is measurable and assumes only a finite number of values. If \( \varphi \) is simple and has the values \( \alpha_1, \ldots, \alpha_n \) then \( \varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \), where \( A_i = \{x: \varphi(x) = \alpha_i\} \). The sum, product, and difference of two simple functions are simple.

\section*{Problems}

18. Show that (v) does not imply (iv) in Proposition 18 by constructing a function \( f \) such that \( \{x: f(x) > 0\} = E \), a given nonmeasurable set, and such that \( f \) assumes each value at most once.

19. Let \( D \) be a dense set of real numbers, that is, a set of real numbers such that every interval contains an element of \( D \). Let \( f \) be an extended real-valued function on \( \mathbb{R} \) such that \( \{x: f(x) > \alpha\} \) is measurable for each \( \alpha \in D \). Then \( f \) is measurable.

20. Show that the sum and product of two simple functions are simple. Show that

\[
\chi_{A \cap B} = \chi_A \cdot \chi_B
\]
\[
\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B
\]
\[
\chi_A = 1 - \chi_A.
\]

21. a. Let \( D \) and \( E \) be measurable sets and \( f \) a function with domain \( D \cup E \). Show that \( f \) is measurable if and only if its restrictions to \( D \) and \( E \) are measurable.

b. Let \( f \) be a function with measurable domain \( D \). Show that \( f \) is measurable iff the function \( g \) defined by \( g(x) = f(x) \) for \( x \in D \) and \( g(x) = 0 \) for \( x \notin D \) is measurable.
except on a set of measure less than \( \epsilon \); i.e., \( m\{x: |f(x) - g(x)| \geq \epsilon \} < \epsilon \) and \( m\{x: |f(x) - h(x)| \geq \epsilon \} < \epsilon \). If in addition \( m \leq f \leq M \), then we may choose the functions \( g \) and \( h \) so that \( m \leq g \leq M \) and \( m \leq h \leq M \).

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\chi_A = 1 - \chi_A.
\]

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except on a set of measure less than \( \epsilon \); i.e., \( m \{ x : |f(x) - g(x)| \geq \epsilon \} < \epsilon \) and \( m \{ x : |f(x) - h(x)| \geq \epsilon \} < \epsilon \). If in addition \( m \leq f \leq M \), then we may choose the functions \( g \) and \( h \) so that \( m \leq g \leq M \) and \( m \leq h \leq M \).

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### Problems

18. Show that (v) does not imply (iv) in Proposition 18 by constructing a function \( f \) such that \( \{ x : f(x) > 0 \} = E \), a given nonmeasurable set, and such that \( f \) assumes each value at most once.

19. Let \( D \) be a dense set of real numbers, that is, a set of real numbers such that every interval contains an element of \( D \). Let \( f \) be an extended real-valued function on \( \mathbb{R} \) such that \( \{ x : f(x) > \alpha \} \) is measurable for each \( \alpha \in D \). Then \( f \) is measurable.

20. Show that the sum and product of two simple functions are simple. Show that

\[
\chi_A \cap B = \chi_A \cdot \chi_B \\
\chi_A \cup B = \chi_A + \chi_B - \chi_A \cdot \chi_B \\
\chi_A^c = 1 - \chi_A.
\]

21. a. Let \( D \) and \( E \) be measurable sets and \( f \) a function with domain \( D \cup E \). Show that \( f \) is measurable if and only if its restrictions to \( D \) and \( E \) are measurable.

b. Let \( f \) be a function with measurable domain \( D \). Show that \( f \) is measurable iff the function \( g \) defined by \( g(x) = f(x) \) for \( x \in D \) and \( g(x) = 0 \) for \( x \notin D \) is measurable.
HWk 3 For Thu Sept 20

Circled probs.
22. a. Let $f$ be an extended real-valued function with measurable domain $D$, and let $D_1 = \{x: f(x) = -\infty\}, D_2 = \{x: f(x) = \infty\}$. Then $f$ is measurable if and only if $D_1$ and $D_2$ are measurable and the restriction of $f$ to $D \sim (D_1 \cup D_2)$ is measurable.

b. Prove that the product of two measurable extended real-valued functions is measurable.

c. If $f$ and $g$ are measurable extended real-valued functions and $\alpha$ a fixed number, then $f + g$ is measurable if we define $f + g$ to be $\alpha$ whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.

d. Let $f$ and $g$ be measurable extended real-valued functions that are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

23. Prove Proposition 22 by establishing the following lemmas:

a. Given a measurable function $f$ on $[a, b]$ that takes the values $\pm \infty$ only on a set of measure zero, and given $\varepsilon > 0$, there is an $M$ such that $|f| \leq M$ except on a set of measure less than $\varepsilon/3$.

b. Let $f$ be a measurable function on $[a, b]$. Given $\varepsilon > 0$ and $M$, there is a simple function $\varphi$ such that $|f(x) - \varphi(x)| < \varepsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take $\varphi$ so that $m \leq \varphi \leq M$.

c. Given a simple function $\varphi$ on $[a, b]$, there is a step function $g$ on $[a, b]$ such that $g(x) = \varphi(x)$ except on a set of measure less than $\varepsilon/3$. [Hint: Use Proposition 15.] If $m \leq \varphi \leq M$, then we can take $g$ so that $m \leq g \leq M$.

d. Given a step function $g$ on $[a, b]$, there is a continuous function $h$ such that $g(x) = h(x)$ except on a set of measure less than $\varepsilon/3$. If $m \leq g \leq M$, then we may take $h$ so that $m \leq h \leq M$.

24. Let $f$ be measurable and $B$ a Borel set. Then $f^{-1}[B]$ is a measurable set. [Hint: The class of sets for which $f^{-1}[E]$ is measurable is a $\sigma$-algebra.]

25. Show that if $f$ is a measurable real-valued function and $g$ a continuous function defined on $(-\infty, \infty)$, then $g \circ f$ is measurable.

26. Borel measurability. A function $f$ is said to be Borel measurable if for each $\alpha$ the set $\{x: f(x) > \alpha\}$ is a Borel set. Verify that Propositions 18 and 19 and Theorem 20 remain valid if we replace "measurable set" by "Borel set" and "(Lebesgue) measurable" by "Borel measurable." Every Borel measurable function is Lebesgue measurable. If $f$ is Borel measurable and $B$ is a Borel set, then $f^{-1}[B]$ is a Borel set. If $f$ and $g$ are Borel measurable, so is $f \circ g$. If $f$ is Borel measurable and $g$ is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

27. How much of the preceding problem can be carried out if we replace the class $\mathcal{B}$ of Borel sets by an arbitrary $\sigma$-algebra $\mathcal{G}$ of sets?

28. Let $f_1$ be the Cantor ternary function (cf. Problem 2.48), and define $f$ by $f(x) = f_1(x) + x$. 
Extend \( \varphi_n \) to all of \( E \) by setting \( \varphi_n(x) = n \) if \( f(x) > n \). The function \( \varphi_n \) is a simple function defined on \( E \) and \( 0 \leq \varphi_n \leq f \) on \( E \). We claim that the sequence \( \{\psi_n\} \) converges to \( f \) pointwise on \( E \). Let \( x \) belong to \( E \).

Case 1: Assume \( f(x) \) is finite. Choose a natural number \( N \) for which \( f(x) < N \). Then

\[
0 \leq f(x) - \varphi_n(x) < 1/n \quad \text{for } n \geq N,
\]

and therefore \( \lim_{n \to \infty} \psi_n(x) = f(x) \).

Case 2: Assume \( f(x) = \infty \). Then \( \varphi_n(x) = n \) for all \( n \), so that \( \lim_{n \to \infty} \varphi_n(x) = f(x) \).

By replacing each \( \varphi_n \) with \( \max\{\varphi_1, \ldots, \varphi_n\} \) we have \( \{\varphi_n\} \) increasing.

PROBLEMS

12. Let \( f \) be a bounded measurable function on \( E \). Show that there are sequences of simple functions on \( E \), \( \{\varphi_n\} \) and \( \{\psi_n\} \), such that \( \{\varphi_n\} \) is increasing and \( \{\psi_n\} \) is decreasing and each of these sequences converges to \( f \) uniformly on \( E \).

13. A real-valued measurable function is said to be semisimple provided it takes only a countable number of values. Let \( f \) be any measurable function on \( E \). Show that there is a sequence of semisimple functions \( \{f_n\} \) on \( E \) that converges to \( f \) uniformly on \( E \).

14. Let \( f \) be a measurable function on \( E \) that is finite a.e. on \( E \) and \( m(E) < \infty \). For each \( \epsilon > 0 \), show that there is a measurable set \( F \) contained in \( E \) such that \( f \) is bounded on \( F \) and \( m(E \sim F) < \epsilon \).

15. Let \( f \) be a measurable function on \( E \) that is finite a.e. on \( E \) and \( m(E) < \infty \). Show that for each \( \epsilon > 0 \), there is a measurable set \( F \) contained in \( E \) and a sequence \( \{\varphi_n\} \) of simple functions on \( E \) such that \( \varphi_n \to f \) uniformly on \( F \) and \( m(E \sim F) < \epsilon \). (Hint: See the preceding problem.)

16. Let \( I \) be a closed, bounded interval and \( E \) a measurable subset of \( I \). Let \( \epsilon > 0 \). Show that there is a step function \( h \) on \( I \) and a measurable subset \( F \) of \( I \) for which

\[
h = \chi_E \text{ on } F \text{ and } m(I \sim F) < \epsilon.
\]

(Hint: Use Theorem 12 of Chapter 2.)

17. Let \( I \) be a closed, bounded interval and \( \psi \) a simple function defined on \( I \). Let \( \epsilon > 0 \). Show that there is a step function \( h \) on \( I \) and a measurable subset \( F \) of \( I \) for which

\[
h = \psi \text{ on } F \text{ and } m(I \sim F) < \epsilon.
\]

(Hint: Use the fact that a simple function is a linear combination of characteristic functions and the preceding problem.)

18. Let \( I \) be a closed, bounded interval and \( f \) a bounded measurable function defined on \( I \). Let \( \epsilon > 0 \). Show that there is a step function \( h \) on \( I \) and a measurable subset \( F \) of \( I \) for which

\[
|h - f| < \epsilon \text{ on } F \text{ and } m(I \sim F) < \epsilon.
\]

19. Show that the sum and product of two simple functions are simple as are the max and the min.
Extend \( \varphi_n \) to all of \( E \) by setting \( \varphi_n(x) = n \) if \( f(x) > n \). The function \( \varphi_n \) is a simple function defined on \( E \) and \( 0 \leq \varphi_n \leq f \) on \( E \). We claim that the sequence \( \{ \psi_n \} \) converges to \( f \) pointwise on \( E \). Let \( x \) belong to \( E \).

Case 1: Assume \( f(x) \) is finite. Choose a natural number \( N \) for which \( f(x) < N \). Then

\[
0 \leq f(x) - \varphi_n(x) < 1/n \text{ for } n \geq N,
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and therefore \( \lim_{n \to \infty} \psi_n(x) = f(x) \).

Case 2: Assume \( f(x) = \infty \). Then \( \varphi_n(x) = n \) for all \( n \), so that \( \lim_{n \to \infty} \varphi_n(x) = f(x) \).

By replacing each \( \varphi_n \) with \( \max(\varphi_1, \ldots, \varphi_n) \) we have \( \{ \varphi_n \} \) increasing.

\[ \square \]

PROBLEMS

12. Let \( f \) be a bounded measurable function on \( E \). Show that there are sequences of simple functions on \( E \), \( \{ \varphi_n \} \) and \( \{ \psi_n \} \), such that \( \{ \varphi_n \} \) is increasing and \( \{ \psi_n \} \) is decreasing and each of these sequences converges to \( f \) uniformly on \( E \).

13. A real-valued measurable function is said to be semisimple provided it takes only a countable number of values. Let \( f \) be any measurable function on \( E \). Show that there is a sequence of semisimple functions \( \{ f_n \} \) on \( E \) that converges to \( f \) uniformly on \( E \).

14. Let \( f \) be a measurable function on \( E \) that is finite a.e. on \( E \) and \( m(E) < \infty \). For each \( \epsilon > 0 \), show that there is a measurable set \( F \) contained in \( E \) such that \( f \) is bounded on \( F \) and \( m(E \setminus F) < \epsilon \).

15. Let \( f \) be a measurable function on \( E \) that is finite a.e. on \( E \) and \( m(E) < \infty \). Show that for each \( \epsilon > 0 \), there is a measurable set \( F \) contained in \( E \) and a sequence \( \{ \varphi_n \} \) of simple functions on \( E \) such that \( \{ \varphi_n \} \to f \) uniformly on \( F \) and \( m(E \setminus F) < \epsilon \). (Hint: See the preceding problem.)

16. Let \( I \) be a closed, bounded interval and \( E \) a measurable subset of \( I \). Let \( \epsilon > 0 \). Show that there is a step function \( h \) on \( I \) and a measurable subset \( F \) of \( I \) for which

\[
h = \chi_E \text{ on } F \text{ and } m(I \setminus F) < \epsilon.
\]

(Hint: Use Theorem 12 of Chapter 2.)

17. Let \( I \) be a closed, bounded interval and \( \psi \) a simple function defined on \( I \). Let \( \epsilon > 0 \). Show that there is a step function \( h \) on \( I \) and a measurable subset \( F \) of \( I \) for which

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(Hint: Use the fact that a simple function is a linear combination of characteristic functions and the preceding problem.)

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\[
|h - f| < \epsilon \text{ on } F \text{ and } m(I \setminus F) < \epsilon.
\]

19. Show that the sum and product of two simple functions are simple as are the max and the min.
Hwk 4

for presentation Sept 27.

Exam 1 on chap. 2 + 3
will be held 1st week in Oct.
Section 3.3  Littlewood's Three Principles, Egoroff's, and Lusin's Theorem

\[
m(E \sim F) = m\left( [E \sim F_0] \cup \bigcup_{n=1}^{\infty} [E \sim F_n] \right) \leq \epsilon/2 + \sum_{n=1}^{\infty} \epsilon/2^{n+1} = \epsilon.
\]

The set \( F \) is closed since it is the intersection of closed sets. Each \( f_n \) is continuous on \( F \) since \( F \subseteq F_n \) and \( f_n = g_n \) on \( F_n \). Finally, \( \{f_n\} \) converges to \( f \) uniformly on \( F \) since \( F \subseteq F_0 \). However, the uniform limit of continuous functions is continuous, so the restriction of \( f \) to \( F \) is continuous on \( F \). Finally, there is a continuous function \( g \) defined on all of \( \mathbb{R} \) whose restriction to \( F \) equals \( f \) (see Problem 25). This function \( g \) has the required approximation properties.

**PROBLEMS**

25. Suppose \( f \) is a function that is continuous on a closed set \( F \) of real numbers. Show that \( f \) has a continuous extension to all of \( \mathbb{R} \). This is a special case of the forthcoming Tietze Extension Theorem. (Hint: Express \( \mathbb{R} \sim F \) as the union of a countable disjoint collection of open intervals and define \( f \) to be linear on the closure of each of these intervals.)

26. For the function \( f \) and the set \( F \) in the statement of Lusin’s Theorem, show that the restriction of \( f \) to \( F \) is a continuous function. Must there be any points at which \( f \), considered as a function on \( E \), is continuous?

27. Show that the conclusion of Egoroff’s Theorem can fail if we drop the assumption that the domain has finite measure.

28. Show that Egoroff’s Theorem continues to hold if the convergence is pointwise a.e. and \( f \) is finite a.e.

29. Prove the extension of Lusin’s Theorem to the case that \( E \) has infinite measure.

30. Prove the extension of Lusin’s Theorem to the case that \( f \) is not necessarily real-valued, but may be finite a.e.

31. Let \( \{f_n\} \) be a sequence of measurable functions on \( E \) that converges to the real-valued \( f \) pointwise on \( E \). Show that \( E = \bigcup_{k=1}^{\infty} E_k \), where for each index \( k \), \( E_k \) is measurable, and \( \{f_n\} \) converges uniformly to \( f \) on each \( E_k \) if \( k > 1 \), and \( m(E_1) = 0 \).
PROBLEMS

9. Let $E$ have measure zero. Show that if $f$ is a bounded function on $E$, then $f$ is measurable and $\int_E f = 0$.

10. Let $f$ be a bounded measurable function on a set of finite measure $E$. For a measurable subset $A$ of $E$, show that $\int_A f = \int_E f \cdot \chi_A$.

11. Does the Bounded Convergence Theorem hold for the Riemann integral?

12. Let $f$ be a bounded measurable function on a set of finite measure $E$. Assume $g$ is bounded and $f = g$ a.e. on $E$. Show that $\int_E f = \int_E g$.

13. Does the Bounded Convergence Theorem hold if $m(E) < \infty$ but we drop the assumption that the sequence $\{|f_n|\}$ is uniformly bounded on $E$?

14. Show that Proposition 8 is a special case of the Bounded Convergence Theorem.

15. Verify the assertions in the last Remark of this section.

16. Let $f$ be a nonnegative bounded measurable function on a set of finite measure $E$. Assume $\int_E f = 0$. Show that $f = 0$ a.e. on $E$.

4.3 THE LEbesgue INTEGRAL OF A MEASURABLE NONnegative FUNCTION

A measurable function $f$ on $E$ is said to vanish outside a set of finite measure provided there is a subset $E_0$ of $E$ for which $m(E_0) < \infty$ and $f = 0$ on $E \sim E_0$. It is convenient to say that a function that vanishes outside a set of finite measure has finite support and define its support to be $\{x \in E \mid f(x) \neq 0\}$. In the preceding section, we defined the integral of a bounded measurable function $f$ over a set of finite measure $E$. However, even if $m(E) = \infty$, if $f$ is bounded and measurable on $E$ but has finite support, we can define its integral over $E$ by

$$\int_E f = \int_{E_0} f,$$

where $E_0$ has finite measure and $f = 0$ on $E \sim E_0$. This integral is properly defined, that is, it is independent of the choice of set of finite measure $E_0$ outside of which $f$ vanishes. This is a consequence of the additivity over domains property of integration for bounded measurable functions over a set of finite measure.

**Definition** For $f$ a nonnegative measurable function on $E$, we define the integral of $f$ over $E$ by

$$\int_E f = \sup \left\{ \int_E h \mid h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E \right\}. \quad (8)$$

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6But care is needed here. In the study of continuous real-valued functions on a topological space, the support of a function is defined to be the closure of the set of points at which the function is nonzero.

7This is a definition of the integral of a nonnegative extended real-valued measurable function; it is not a definition of what it means for such a function to be integrable. The integral is defined regardless of whether the function is bounded or the domain has finite measure. Of course, the integral is nonnegative since it is defined to be the supremum of a set of nonnegative numbers. But the integral may be equal to $\infty$, as it is, for instance, for a nonnegative measurable function that takes a positive constant value of a subset of $E$ of infinite measure or the value $\infty$ on a subset of $E$ of positive measure.
Section 4.3  The Lebesgue Integral of a Measurable Nonnegative Function  

PROBLEMS

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4.3 THE LEBESGUE INTEGRAL OF A MEASURABLE NONNEGATIVE FUNCTION

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HWK 6 (HWks 5, 6 + 7) for test 2
in Late Oct

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48, 50