Integral Transform: Suppose that $X$ is continuously distributed and $h(x)$ is a nice function; We expect $h(X) = Y$ to be a random variable, and its density $g_Y(y)$ is:

First calculate its distribution function $G_Y(a) = P(Y < a) = P(h(X) < a)$

For increasing $h(x)$, we can find the inverse $X < h^{-1}(a)$, and then $G_Y(a) = F_X(h^{-1}(a))$

For decreasing $h(x)$, again we can rewrite the event $\{h(X) < a\}$ as $\{X > h^{-1}(a)\}$; Then $G_Y(a) = 1 - F_X(h^{-1}(a))$

Finally, differentiate to get: $g_Y(y) = \frac{d}{dy} G_Y = f_X(h^{-1}(y)) \cdot \left| \frac{d}{dy} h^{-1}(y) \right|$
APPLICATIONS TO NON MONOTONE CASE

Consider the rv \( X \) which is uniformly distributed in \((-1, 2)\) or \( X \) is U(-1,2):

(a) Calculate density function, \( f_X(x) = 1/3 \)  Ex

(b) Calculate the density function of \( Y = X^2 \): \( g_Y(y) = ?? \)

Ans: Note that the range of \( Y \) is \((0, 4)\); then \( G_Y(a) = P(Y < a) = P(\{X^2 < a\})\);

For \( 0 < a < 1 \), \( \{X^2 < a\} = \{- \sqrt{a} < X < \sqrt{a}\} \) and \( P(\{- \sqrt{a} < X < \sqrt{a}\}) = 2\sqrt{a} / 3. \) Ex

For \( 1 < a < 4 \), \( \{X^2 < a\} = \{X < \sqrt{a}\} \) and \( P(\{X < \sqrt{a}\}) = (\sqrt{a} + 1)/3. \) Ex

Then \( g_Y(y) = d/dy G_Y(a) = 1/(3\sqrt{y}) \) for \( 0 < y < 1. \) Ex. Calculate it for \( 1 < y < 4 ?? \)
Theorem: Let X be a continuous rv with distribution function $F(a) = F_X(a) = P(X < a)$. Then $Y = F(X)$ is uniformly distributed $U(0,1)$, i.e., $g_Y(y) = 1$ if $y$ is in $(0,1)$, and $g_Y(y) = 0$ if $y$ is not in $(0,1)$,

Proof: $G_Y(b) = P(Y < b) = P(F(X) < b) = P(X < F^{-1}(b))$ if $F$ monotone increasing. If $F$ not strictly increasing, that is, for $a < c$, $F(a) = F(c)$, then take $F^{-1}(b) = \sup \{x \mid F(x) < b\}$.

Then $G_Y(b) = P(X < F^{-1}(b)) = F(F^{-1}(b)) = b$. QED

Ex: Why is $G_Y(b) = b$ for all $b$ in $(0,1)$ indicative of a $U(0,1)$ distribution?
Read example 4.19 upto and including theorem 4. 22:

Theorem: Let $X$ and $Y$ be independent rv’s with density functions $f_X$ and $f_Y$. Then, $Z = X + Y$ has density

$$f_Z(z) = \text{convolution of } f_X \text{ and } f_Y.$$ 

Proof: The joint density function of $X$ and $Y$ is $f_{X,Y}(x,y) = f_X(x) f_Y(y)$. Thus, the distribution function of $Z = X + Y$ is

$$F_Z(t) = P\{Z < t\}$$

$$= \text{double integral over } \{X + Y < t\} \left[ f_X(x) f_Y(y) \right]$$

$$= \text{integral over reals } [f_X(x) \, dx] \times \text{integral from } -\infty \text{ to } (t-x) \times [f_Y(y) \, dy]$$

$$= \text{integral over reals } [f_X(x) \, F_Y(t-x) \, dx]$$

because integral from $-\infty$ to $(t-x)$ $[f_Y(y) \, dy] = F_Y(t-x)$ by definition.
Theorem: Suppose $X$ and $Y$ are jointly distributed with joint density $f_{X,Y}(x,y)$. Let $u = u(x,y)$, $v = v(x,y)$ be invertible mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. And let $U = u(X,Y)$, $V = v(X,Y)$ be jointly distributed rv’s with density $g_{U,V}(u,v)$. Then $g_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \cdot |J(u,v)|$.

Where $|J(u,v)|$ is the Jacobian of the inverse mapping $x = x(u,v)$, $y = y(u,v)$.

Proof:

Recall the change of variables formula from Multivar Calc: from definition, for arbitrary event $A$, $P\{(U,V) \in A\} = \int_{A} g_{U,V}(u,v) \, dudv = P\{(X,Y) \in B = (u,v)^{-1}(A)\}$ or

$$\int_{A} g_{U,V}(u,v) \, dudv = \int_{B} f_{X,Y}(x,y) \, dxdy$$

Since $A$ arbitrary,

$g_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \cdot |J(u,v)|$.