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## Preface

This book is meant for an audience of advanced undergraduates and graduate students taking courses on the statistical mechanics approach to turbulent flows and on stochastic simulations. It is also suitable for the self-study of professionals involved in the research and modelling of large scale stochastic fluid flows with a substantial vortical component.

Several related ideas motivate the approach in this book, namely, the application of equilibrium statistical mechanics to two-dimensional and 2.5-dimensional fluid flows in the spirit of Onsager [337], and Kraichnan [227], is taken to be a valid starting point, and the primary importance of non-linear convection effects combined with the gravitational and rotational properties of large scale stratified flows over the secondary effects of viscosity is assumed. The latter point is corroborated by the many successful studies of fluid viscosity which limit its effects to specific and narrow regions such as boundary layers, and to the initial and transient phases of the experiment such as in the Ekman layer and spin-up [154] [344].

The main point of applying equilibrium statistical methods to the problems in this book is underscored by the values of the Knudsen number  $K = \lambda/l$  (where  $\lambda$  is the mean free path of the molecules of the fluid and  $l$  is the smallest relevant macroscopic length scale in the flow) in the body of two-dimensional and 2.5-dimensional large scale fluid flows treated here, namely  $K < 10^{-6}$ . We further elucidate this point by stressing the fact that in this book, the methods of statistical mechanics are applied not to the fluid as an ensemble of molecules but rather to the flow as an ensemble of vorticity parcels. Nonetheless, many of the techniques used in the statistical treatment of molecular thermodynamics, including the spin-lattice models pioneered in the study of magnetism in condensed matter physics, can be adapted for our primary purpose here.

Our approach of applying equilibrium statistical mechanics to vortical flows centers on the extremization of the free energy  $F = U - TS$  where  $U$  is the internal energy and  $S$  is the entropy. Besides the standard application of Planck's theorem to thermal systems at positive temperatures, where

one minimizes the free energy, we are also interested in vortex problems at negative temperatures, where one maximizes the free energy to obtain the thermodynamically stable statistical equilibria. This point is explored in a simple mean field theory for barotropic flows on a rotating sphere that relates for the first time positive and negative critical temperatures of phase transitions to the key variables of planetary spin, relative enstrophy and kinetic energy.

We note that at low enough positive temperatures  $T$ , the minimization of  $F$  can be profitably approximated by the easier ground state problem. We further note that the ground state problem by virtue of the minimization of augmented energy functionals, is directly related to steady-state flows of the associated Euler equations. Finally, these special steady-states are related back to decaying Navier-Stokes flows by the Principle of Selective Decay or Minimum Enstrophy, which states that the Dirichlet Quotient (defined as enstrophy over energy) in many damped two-dimensional viscous flows tends asymptotically to a minimum, achieved by the special steady-states.

Vortex statistics is special not only because negative temperatures occur at high energies (a curious phenomenon we will explain in detail), but because it is also characterized by the wide range of temperatures over which extremals of the free energy are close to the corresponding extremals of the internal energy. We will explore the physical reasons for these interesting phenomena in several archetypical examples of vortex dynamics. The most important of these problems are the crystalline or polyhedral equilibria of  $N$  point vortices on the sphere, the thermal equilibria of the Onsager vortex gas on the unbounded plane with respect to dynamical equilibria of the rotating two-dimensional Euler equations, and the thermal equilibria at negative temperatures of barotropic vortex dynamics on a rotating sphere.

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