Negative specific heat in a quasi-2D generalized vorticity model

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(Dated: June 18, 2007)

Negative specific heat is a dramatic phenomenon where processes decrease in temperature when adding energy. It has been observed in gravo-thermal collapse of globular clusters. We now report finding this phenomenon in bundles of nearly parallel, periodic, single-sign generalized vortex filaments in the electron magnetohydrodynamic (EMH) model for the unbounded plane under strong magnetic confinement. We derive the specific heat using a steepest descent method and a mean field property. Our derivations show that as temperature increases, the overall size of the system increases exponentially and the energy drops. The implication of negative specific heat is a runaway reaction, resulting in a collapsing inner core surrounded by an expanding halo of filaments.

PACS numbers: 47.27.jV, 47.32.cb, 52.25.Xz, 52.35.Ra

Negative specific heat is an unusual phenomenon first discovered in statistical models of gravo-thermal collapse in globular clusters [11]. Since then it has been observed in very few other places. Its counterintuitive implication is that as a system loses energy its temperature increases and as one adds energy the system cools. In globular clusters, this results in a runaway reaction leading to a collapse of the cluster. In a magnetic fusion system or other thermally isolated plasma, should negative specific heat exist, the related runaway collapse could have profound implications for fusion where extreme confinement is critical to a sustained reaction.

In this letter we address negative specific heat in the electron magnetohydrodynamical (EMH) model for plasmas under strong angular momentum confinement in the plane. The EMH model bypasses the complexities of the two-fluid, magnetohydrodynamical model by representing the electron fluid and the magnetic field as a single, generalized fluid with a neutralizing background. This model takes the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, and the charged fluid vorticity, $\omega = \nabla \times \mathbf{v}$, and combines them into a general vorticity field $\Omega = \nabla \times \mathbf{p}$ where the generalized momentum, $\mathbf{p} = m\mathbf{v} - e\mathbf{A}$, m is the electron mass, $-e$ is the electron charge, $\mathbf{v}$ is the fluid velocity field, and $\mathbf{A}$ is the magnetic vector potential field. The EMH model statistics have the advantage of being mathematically tractable in the first order nearly parallel filament case given that we make some mean-field assumptions. For a brief overview of the model, see [14]. A detailed model discussion can be found in [3].

Our statistical mechanical approach requires a discrete vorticity field. Therefore, we assume that Ω is made up of a large number of periodic (assume a period of 1 without loss of generality) filaments that are nearly parallel to the z-axis,

$$\Omega(\mathbf{r}) = \sum_{i=1}^{N} \int_0^1 d\tau \Gamma_i \delta(\mathbf{r} - \mathbf{r}_i(\tau)), \qquad (1)$$

where N is the number of filaments, $\mathbf{r} = (x, y, z)$ and $\mathbf{r}_i = (x_i, y_i, z_i)$. Periodicity requires that $\mathbf{r}_i(0) = \mathbf{r}_i(1)$. Because of the nearly parallel constraint the arclength $\tau \sim z_i \forall i$. For simplicity we will represent $\mathbf{r}_i(\tau) = (x_i, y_i, \tau)$ as a complex number $\psi_i(\tau) = x_i(\tau) + iy_i(\tau)$. For computational approaches discretization is a necessary step, but, even in our analytical approach, it is easier to start with a finite number of filaments and take the non-extensive thermodynamic limit, $N \to \infty$, later. In order to accomplish this, we employ the spherical methods of [2, 4] to address the microcanonical ensemble in the thermodynamic limit. These methods were first applied in a vorticity model on a barotropic sphere [9],[7]. We believe ours to be the first application to a vortex filament model.

Since magnetically confined plasmas should be isolated with conserved angular momentum, the correct model is microcanonical in enthalpy and angular momentum:

$$P(s) = Z^{-1} \delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N), \qquad (2)$$

where H_0 is the total "enthalpy" per vortex per period of the plasma and $Z = \int ds \delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N)$. Here E_N is the energy functional and M_N is the angular momentum. It is our intent to allow R^2 to be determined by other parameters in the system and keep enthalpy and pressure, p, fixed.

We take a mean-field approach making all filaments statistically independent. We treat the logarithmic interaction between vortices as an interaction between a perfectly straight filament a mean distance from the origin and the statistical center of charge of all the filaments, which is a single, perfectly straight filament fixed at the origin with strength $N - 1 \sim N$. Mathematically, this allows us to replace the distance between a filament i and a filament j with the $\mathcal{L}_2$-norm of the curve of filament i

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(or j),

$$\langle|\psi_i - \psi_j|\rangle \longrightarrow \sqrt{\int_0^1 d\tau|\psi_i(\tau)|^2} = ||\psi_i||, \qquad (3)$$

where i is any filament index and the double bars $||\cdot||$ indicate $\mathcal{L}_2$-norm on the interval $[0,1]$. This distance-based assumption follows from the axisymmetry of the problem, and Monte Carlo simulations support its validity [8]. With it we derive an explicit formula for the maximal entropy of the thermally isolated system of filaments in the non-extensive limit, from which we derive the specific heat.

While [13] has proven that systems canonical in energy must have positive specific heat, the specific heat in energetically isolated systems can be negative [10]. We now report finding negative specific heat in the thermally microcanonical system of confined generalized vorticity filaments, a finding of great interest to magnetohydrodynamics (MHD) and plasma physics.

Heuristically, because the vortices are nearly parallel, we can combine two results, that of [6] that shows that a discrete, MHD field of perfectly parallel vortices has a logarithmic interaction, and that of [14] that shows a first order local self-induction for the motion of a single filament that depends on the binormal of the curve—essentially Brownian variations in the vortex filament. In first order, this combination yields the familiar London free energy common to type-II superconductors [12] but equally valid for generalized vorticity [14]:

$$\begin{aligned} E_N =& \alpha \int_0^1 d\tau \sum_{i=1}^{N} \Gamma_i/2 \left|\partial\psi_i(\tau)/\partial\tau\right|^2 - \\ & \int_0^1 d\tau \sum_{i=1}^{N}\sum_{j>i}^{N} \Gamma_i\Gamma_j \log|\psi_i(\tau) - \psi_j(\tau)|, \end{aligned} \qquad (4)$$

where α is the vortex elasticity in units of energy/length, Γ_i is the vortex circulation. From this point on, we assume the vortex circulations are all scaled to unity, $\Gamma_i = 1 \forall i$. Although we say this is the London free energy, we note that it is not describing the same type of system that it would describe in the type-II superconductor setting because the vortex filaments are generalized and not flux lines. What makes this system perhaps even more interesting is that it exists in an unbounded plane rather than with periodic boundary conditions, doing away with artificial planar periods. The equilibrium length scale of this system is completely determined by the maximal entropy or most-probable macrostate.

The size of the configuration space (partition function) Z cannot be found in closed form by any known analytical methods. Since our goal is an explicit expression for the Helmholtz free energy, we need to know Z in closed form. Therefore, we make a reasonable mean-field assumption on the energy functional,

$$E'_N = \int_0^1 d\tau \sum_{i=1}^{N} \left[\frac{\alpha}{2}\left|\frac{\partial\psi_i}{\partial\tau}\right|^2 - \frac{N}{4}\log||\psi_i||^2\right]. \qquad (5)$$

This assumption changes the interaction between vortices from fully coupled to fully decoupled, that is, all vortices are statistically independent, and the statistics of the entire system can be found from the statistical behavior of a single vortex.

The probability distribution over microstates s is now

$$P'(s) = Z^{-1}\delta(NE_0 - E'_N - pM_N)\delta\left(NR^2 - M_N\right). \qquad (6)$$

However, we leave R to be determined by the entropy maximization rather than specifying its value since the energy, E_0, completely determines R. For the rest of this letter, we drop the prime on E'_N.

We now proceed to derive an explicit formula for the maximal entropy of this system in the non-extensive thermodynamic limit. We refer the reader to [5] for the following procedure: Given our previous assumptions about the enthalpy, the entropy per filament, S_N, is defined by,

$$e^{S_N} = \int D\psi\delta(NH_0 - E_N - pNR^2)\delta(NR^2 - M_N). \qquad (7)$$

This definition implies,

$$S_N = \log\left[\int D\psi\delta(NH_0 - E_N - pNR^2)\delta(NR^2 - M_N)\right]. \qquad (8)$$

We need to find R^2 as a minimization of the canonical free enthalpy. Replacing the first delta function with its integral representation yields,

$$e^{S_N} = \int D\psi \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{d\beta}{2\pi i} e^{\beta NH_0 - \beta E_N - \beta NR^2}\delta(NR^2 - M_N). \qquad (9)$$

Given the obviously finite result above, we can rewrite the entropy relation as,

$$\begin{aligned} e^{S_N} &= \int \frac{d\beta}{2\pi i} e^{\beta NH_0} \int D\psi e^{-\beta(E_N + pNR^2)}\delta(NR^2 - M_N) \\ &= \int \frac{d\beta}{2\pi i} e^{\beta NH_0} Z_c \end{aligned} \qquad (10)$$

where

$$Z_c = \int D\psi e^{-\frac{N\alpha\beta}{2}\left|\frac{\partial\psi}{\partial\tau}\right|^2 - \beta pNR^2 + \beta N^2/4\log R^2}\delta(NR^2 - M_N) \qquad (11)$$

is the partition function for the non-thermally-isolated system of a single filament.

Now we need to take the steepest-descent limit to find the maximal entropy,

$$S_{max}(H_0) = \lim_{N\to\infty} N^{-1}S_N. \qquad (12)$$

An approximation for the free energy as $F = -\lim_{N\to\infty} \frac{1}{N\beta} \log Z_c$ is found in Chapter 4 of Andersen's thesis [1]. We know that, if we make the scaling $\beta' = \beta N$, $\alpha' = \alpha/N$, $p' = p/N$, and $H_0' = H_0/N$, necessary to make the interaction energy grow with N rather than N^2, we have the free energy,

$$F = p'R^2 - 1/4 \log R^2 + \frac{1}{2\alpha'\beta_0'^2 R^2}, \tag{13}$$

where

$$R^2 = \frac{\beta_0'^2\alpha' + \sqrt{\beta_0'^4\alpha'^2 + 32\alpha'\beta_0'^2 p'}}{8\alpha'\beta_0'^2 p'}, \tag{14}$$

minimizes F.

Replacing Z_{can} with $\exp(-\beta' N F)$ (valid for large N), gives

$$e^{NS_N} = \int \frac{d\beta'}{2\pi i} e^{\beta' N H_0' - \beta' N F}, \tag{15}$$

and, because $S_{max} = \lim_{N\to\infty} S_N$,

$$\begin{aligned} S_{max}(H_0) &= \beta_0' H_0' - \beta_0' F \\ &= \beta_0' H_0' + \frac{\beta_0'}{4}\log(R^2) - \frac{1}{2\alpha\beta_0 R^2} - \beta_0' p' R^2. \end{aligned} \tag{16}$$

This entropy is exact within the mean-field assumption for $N \to \infty$.

We find the unknown multiplier, β_0, by relating the enthalpy per filament parameter, H_0, to the mean enthalpy, $NH_0 = \langle E_N + pM_N\rangle$, where $\langle\cdot\rangle$ denotes average against Equation 2.

By definition the average enthalpy is given by

$$\langle E_N + pM_N\rangle = \frac{\int D\psi E_N \delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N)}{\int D\psi \delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N)} \tag{17}$$

We can simplify this equation with steepest-descent methods in the non-extensive limit, following [5]:

$$\begin{aligned} e^{NS_N} &= \int D\psi \delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N) \\ &= \int D\psi \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{d\beta}{2\pi i} e^{\beta N H_0} e^{-\beta E_N - \beta p M_N} \delta(NR^2 - M_N) \end{aligned} \tag{18}$$

As N becomes large, the saddle point method gives

$$\langle E_N + pM_N\rangle \approx N \frac{e^{\beta_0' N H_0'}\left(-\frac{\partial}{\partial\beta_0'} e^{-\beta_0' N F}\right)}{e^{\beta_0' N H_0'} e^{-\beta_0' N F}}, \tag{19}$$

which implies that, because $N^2 H_0' = \langle E_N\rangle$,

$$\begin{aligned} H_0' &= \lim_{N\to\infty} -\frac{1}{N}\frac{\partial e^{-\beta_0' N F}}{\partial \beta_0'}\frac{1}{e^{-\beta_0' N F}} \\ &= \lim_{N\to\infty} -\frac{1}{N}\frac{\partial \log e^{-\beta_0' N F}}{\partial\beta_0'}, \\ &= \frac{\partial}{\partial\beta_0'}\left(-\frac{\beta_0'}{4}\log R^2 + \frac{1}{2\alpha'\beta_0' R^2} + \beta_0' p' R^2\right) \end{aligned} \tag{20}$$

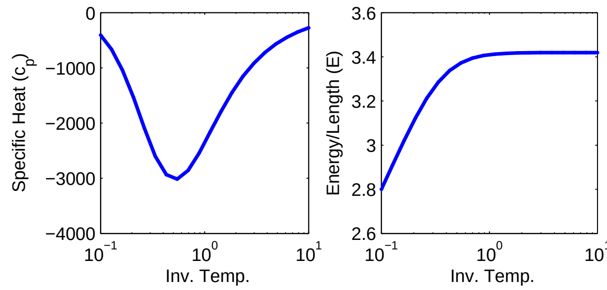

FIG. 1: The specific heat at constant pressure (Equation 22) for the thermally isolated system is negative, meaning that the constant pressure Enthalpy Per Length (Equation 20) decreases with increasing temperature. (Here $\alpha' = 5\times10^5$ and $p' = 8\times10^4$.)

exactly.

The above derivation confirms that we can find the average enthalpy in the thermodynamic limit using the canonical formulation, noting that R^2 is defined by Equation 14. The constants p, α, and β_0 are required to be positive for the model to have a finite partition function. We cannot give an explicit expression for β_0 because it is a root of a transcendental equation, but such is unnecessary for the following negative specific heat result:

We define specific heat at constant generalized pressure, p,

$$c_p = -\beta_0^2\frac{\partial H_0}{\partial\beta_0} = -\beta_0^2\frac{\partial^2\beta_0 F}{\partial\beta_0^2} \tag{21}$$

and after evaluating with Equation 20 and simplifying (dropping primes and 0-subscripts)

$$c_p = \frac{\beta}{4}\left(\frac{\alpha\beta^2}{\sqrt{\alpha\beta^2(\alpha\beta^2 + 32p)}} - 1\right). \tag{22}$$

Equation 22 is significant. It indicates that the specific heat is not only negative for this system, but *strictly* negative if parameters are non-zero (Figure 1). In the low-temperature (large β) case, for constant field strength, R^2 does not change significantly with temperature indicating that filaments are in a stable configuration for a large range of low-temperatures. Because the filaments do not move relative to one another at low-temperatures and the self-induction is negligible, the enthalpy does not change. When temperature becomes high the internal entropy causes a massive expansion in the overall size of the system (Figure 2), and energy of the logarithmic interaction decreases far more than the enthalpy of the self-induction increases. The strong magnetic field absorbs this energy, but, since it is assumed to be an infinitely massive reservoir able to maintain the enthalpy at H_0, the confinement remains constant.

On a side note, because the specific heat in Equation 22 does not cross the axis (i.e. is never positive) for any

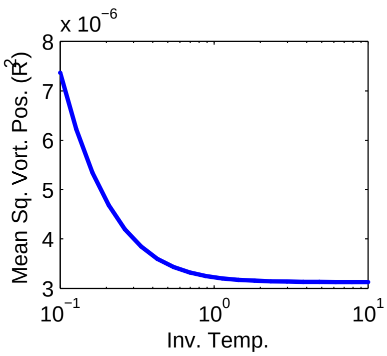

FIG. 2: The Mean Square Vortex Position (Equation 14) increases exponentially at high temperature, while it is nearly constant at low-temperature. (Here $\alpha = 5 \times 10^5$ and $p = 8 \times 10^4$.)

positive parameters, the expansion in R^2 in the canonical system is not likely a phase transition in β, but a continuous "transition" [10]. The system appears to change behavior significantly, but it changes without any discontinuity in the free energy or specific heat.

As mentioned above the negative specific heat indicates a runaway reaction in the thermally microcanonical system that models strongly confined, single-sign generalized vortex filaments. We hypothesize that this kind of runaway reaction, observed in gravo-thermal collapse of globular clusters [11], can lead to two possible outcomes: (1) a collapse similar to globular clusters in which an outer halo of columns separates from an inner core that collapses in on itself, possibly resulting in nuclear fusion, (2) a turbulent expansion of the entire system. Further research will focus on answering this question, but clearly 3D effects are crucial.

Acknowledgments: This work is supported by ARO grant W911NF-05-1-0001 and DOE grant DE-FG02-04ER25616.

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