Anomalous expansion and negative specific heat in quasi-2D plasmas

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Magnetic Nuclear Fusion

Magnetic nuclear fusion is one of the most promising avenues for renewable energy sources.

Figure: The tokamak is a toroidal ring with magnetic fields confining the plasma.
Magnetic Nuclear Fusion continued

\[ \frac{2}{1}D + \frac{3}{1}T \rightarrow \frac{4}{2} \text{He} + \frac{1}{0} \text{n} + 17.6 \text{MeV}. \] (1)

- Plasma fuel (deuterium and tritium) is injected into the torus
- Electric current (about 3.5 MA) heat it to about 100 million K and align it.
- Ring currents confine the plasma to create sufficient central density for fusion.
Instabilities require explanation to be overcome

Confinement can only be sustained for a few seconds.

- Instabilities such as a sawtooth shaped cyclical drop in core temperature prevent sustained fusion.
- Expansion of the central density transports energy from the hot core to the cooler edge where it is lost.
- Instability and expansion are not well explained.
A statistical vortex model

Because of the high number of degrees of freedom in the plasma, we apply a statistical vortex model to explain:

- Anomalous expansion.
- Cyclical instability.
Electron Magnetohydrodynamic (EMH) Model

Electron plasmas.
The EMH model...

- has only one fluid, electrons (40,000 kps), ions (600 kps) are neutralizing background
- treats vorticity as generalized, \( \Omega(r) = \nabla \times p \), where
  \[
  p = mu - eA, \tag{2}
  \]
  is generalized momentum, \( m \) is electron mass, \(-e\) is electron charge, and \( A \) is vector potential for the magnetic field.
- is combination of Navier-Stokes equations and Maxwell’s equations
- implies equations of motion for generalized vorticity are the same as in neutral fluid model

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[Uby et al.(1995)Uby, Isichenko, and Yankov]
The interaction potential between two filaments, \( i \) and \( j \), in any given plane parallel to the \( xy \)-plane is proportional to \( \log(1/r_{ij}) \), where \( r_{ij} \) is the distance between them.
Nearly Parallel Vortex Filaments

In large groups vortex filaments can be treated statistically and simulated on a computer.
Energy and Angular Momentum

Conserved quantities:

Energy

\[ H_N = \alpha \int_0^1 d\tau \sum_{i=1}^N \frac{1}{2} \left| \frac{\partial \psi_i(\tau)}{\partial \tau} \right|^2 - \int_0^1 d\tau \sum_{i=1}^N \sum_{j>i}^N \log |\psi_i(\tau) - \psi_j(\tau)|, \]

where \( \alpha \) is related to the velocity in the core, \( N \) is the number of filaments, and 1 is the period.

Angular Momentum

\[ l_N = \sum_i \int_0^1 d\tau |\psi_i(\tau)|^2. \]

A state \( s = \{\psi_1, \ldots, \psi_N\} \), where \( \psi_j(\tau) = x_j(\tau) + iy_j(\tau), \tau = z, \)

and \( \psi_j(0) = \psi_j(1) \).
The canonical probability distribution applies to all classical physical systems in equilibrium.

For two conserved quantities,

\[
P_c = Z_c^{-1} e^{-\beta H_N - \mu I_N},
\]

(5)

where \(\beta\) and \(\mu\) are constants (\(\mu/\beta\) is akin to pressure, \(\beta = 1/T\) where \(T\) is temperature) and

\[
Z_c = \int D\psi_1 \cdots D\psi_N e^{-\beta H_N - \mu I_N},
\]

(6)

is the normalizing partition function.
Prediction of Monte Carlo

The formula predicts the Monte Carlo Simulated expansion.

Figure: As $\beta$ decreases beyond a threshold, expansion occurs in deviation from 2D point vortex results.
Instability needs a microcanonical (fixed energy) ensemble.

- Assume a continuous density of vortex filaments at a particular cross-section, $g(\vec{x}, \vec{c})$ where $\vec{c} = d\psi/d\tau$.
- Define the entropy by Boltzmann,

$$S = - \int d^2x d^2c \, g \log g, \quad (7)$$

- Take the variation, $\delta S = 0$, with fixed energy, $E = \mathcal{T} + \mathcal{V}$.

$$\mathcal{T} = \frac{\alpha}{2} \int d^4(x, c) \, gc^2, \quad (8)$$

$$\mathcal{V} = \frac{1}{2} \mu' \int d^4(x, c) \, g|\vec{x}|^2 - \frac{1}{\epsilon} \int d^4(x, c) \, d^4(x', c') \, gg' \log |\vec{x} - \vec{x}'|, \quad (9)$$
The variation results in an ODE:

\[
\frac{d^2 v_1}{dz^2} + \frac{1}{z} \frac{dv_1}{dz} + e^{-v_1(z) + \mu z v_1'(z)} = 0, \quad v_1(0) = v_1'(0) = 0, \quad (10)
\]

- Energy,
  \[
  E = \frac{\Lambda^2}{\epsilon} \left( \frac{z^2 e^{-v_1 + \mu z v_1'}}{2(-z v_1')^2} - \frac{1}{(-z v_1')} \right), \quad (11)
  \]

- Temperature,
  \[
  \frac{1}{T} = \beta = -\frac{\epsilon z v_1'(z)}{\Lambda}; \quad (12)
  \]

- Core density,
  \[
  \rho(0) = \frac{\epsilon z^2}{4\pi \beta R^2}, \quad (13)
  \]
Low Confinement ($\mu < 0.5$)

At low confinement (unrealistic) the ensemble has positive specific specific heat.

![Graphs showing Energy, Temperature, and Density vs. z at $\mu=0.4$.]
High Confinement ($\mu > 0.5$)

At high confinement (realistic) the ensemble has negative specific heat.

\begin{align*}
\text{Energy vs. } z \text{ at } \mu = 0.6 \\
\text{T vs. } z \text{ at } \mu = 0.6 \\
\rho_0 \text{ vs. } z \text{ at } \mu = 0.6 \\
\end{align*}
Gravothermal Catastrophe in Globular Clusters

Undergo “core collapse” with overall expansion

Statistical mechanics indicates that a gas of stars has negative specific heat (energy is inversely proportional to temperature), causing the system to be meta-stable. ²

²[Lynden-Bell and Wood(1968)]
Runaway Expansion Mechanism

Instead of a core collapse we have a core expansion.

- Slight expansion occurs and potential energy decreases.
- Kinetic energy increases in response.
- Increase in kinetic energy causes further expansion.
- Cycle continues until a more stable state is reached.
Conclusion

Because like vortices repel rather than attract, the runaway expansion is core collapse in reverse.

- The sawtooth metastability seen in electron core temperature is the result of a runaway expansion.
- This mechanism also explains the anomalous expansion in the canonical ensemble.
- The forcing current creates a cycle.
D. Lynden-Bell and R. Wood. 
The gravo-thermal catastrophe in isothermal spheres and the onset of red-giant structure for stellar systems. 

L. Uby, M. B. Isichenko, and V. V. Yankov. 
Vortex filament dynamics in plasmas and superconductors. 