7. Basic Renewal Theory – Lecture 7 – October 1, 2002

7.1 Basic Renewal Theory

Let \( X_n \) be the time between the \((n-1)\)st and \(n\)th event, or a sequence of random, independent, identically distributed inter-arrival times. let \( \mu = E[X_n] = \int_0^\infty x dF(x) \) be the mean time, where for definiteness we take \( F(0) < 1 \).

Define \( Y_0 = 0 \), and \( Y_n = \sum_{i=1}^n X_i, n \geq 1 \) so that \( Y_n \) is the time of the \(n\)th event. Let \( N(t) \) be the number of events that occur by time \( t \). Then we have the result:

**Proposition 1** \( N(t) = \sup_{n \geq 1} \{ n | Y_n \leq t \} \)

Define the convolution \( F \ast G \) of two distribution functions \( F \) and \( G \) of independent \( X \) and \( Y \) by

\[
(F \ast G)(a) = \text{Prob}\{X + Y \leq a\} = \int_{-\infty}^{\infty} \text{Prob}\{X + Y \leq a | Y = y\} dG(y) = \int_{-\infty}^{\infty} F(a - y) dG(y)
\]

7.2 Properties of \( N(t) \)

**Proposition 2** \( N(t) > n \iff Y_n \leq t \)

This implies that

\[
\text{Prob}\{N(t) = n\} = \text{Prob}\{N(t) \geq n\} - \text{Prob}\{N(t) \geq n + 1\} = \text{Prob}\{Y_n \leq t\} - \text{Prob}\{Y_{n+1} \leq t\}
\]

Since the \( X_k \) are independent identically distributed, with common distribution \( F \), we have that the variables \( \{Y_n = \sum_{i=1}^n X_i\}_{n=1}^{\infty} \) or the \(n\)-fold convolution of \( F \) with itself. This means that

\[
\text{Prob}\{N(t) = n\} = F_n(t) - F_{n+1}(t)
\]
Let \( m(t) \equiv E[N(t)] \) which is called the renewal function.

**Proposition 3** \( m(t) = \sum_{n=1}^{\infty} F_n(t) \)

**Proof.** Let us write \( N(t) = \sum_{n=1}^{\infty} X_n \) where

\[
X_n = \begin{cases} 
1 & \text{if the } n\text{th event is in } [0, 1] \\
0 & \text{otherwise}
\end{cases}
\]  

(7.7)

Then

\[
m(t) = E[N(t)] = E[\sum X_n] = \sum E[X_n] \text{ acceptable from non-negativity of } X_n \]  

(7.8)

\[
= \sum \text{Prob}[X_n = 1] \]  

(7.9)

\[
= \sum \text{Prob}\{Y_n \leq t\} \]  

(7.10)

\[
= \sum_{n=1}^{\infty} F_n(t) \]  

(7.11)

**Proposition 4** \( m(t) < \infty, \forall 0 \leq t < \infty \)

Define \( N(\infty) = \lim_{t \to \infty} N(t) \).

**Proposition 5** \( \text{Prob}\{N(\infty) = \infty\} = 1 \).

**Proof.** The only way for \( N(\infty) \) to be finite is for \( X_n = \infty \) for some \( n < \infty \). So

\[
\text{Prob}\{N(\infty) < \infty\} = \text{Prob}\{X_n = \infty \text{ for } n < \infty\} = \text{Prob}\{\cup_{m=1}^{\infty} \{X_m = \infty\}\} \leq \sum_{n=1}^{\infty} \text{Prob}\{X_n = \infty\} = 0
\]  

(7.13)

(7.14)

(7.15)

(7.16)

(because there is a distribution).

What about the rate at which \( N(t) \) grows infinitely large, i.e., what is the value of \( \lim_{t \to \infty} \frac{N(t)}{t} \)?
Consider \( Y_{N(t)} \) which is the time of the event number \( N(t) \) or the time of the last event prior to (or at) time \( t \). Then \( Y_{N(t)+1} \) is the time of the first event after time \( t \).

**Theorem 1** \( \text{Prob}\{\lim_{t \to \infty} \frac{N(t)}{t} \to \frac{1}{\mu}\} = 1. \)

**Proof.** Observe that

\[
Y_{N(t)} \leq t < Y_{N(t)+1}
\]

by the above discussion. So

\[
\frac{Y_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{Y_{N(t)+1}}{N(t)}
\]

where \( \frac{Y_{N(t)}}{N(t)} \) is the average of the first \( N(t) \) inter-arrival times. By the Strong Law of Large Numbers, we have seen that \( \lim_{n \to \infty} \frac{Y_n}{n} = \mu \). Since \( N(t) \to \infty \) as \( t \to \infty \) we get

\[
\frac{Y_{N(t)}}{N(t)} \to \mu \text{ as } t \to \infty
\]

Now write

\[
\frac{Y_{N(t)+1}}{N(t)} = \left( \frac{Y_{N(t)+1}}{N(t)+1} \right) \left( \frac{N(t)+1}{N(t)} \right) \to_{t \to \infty} \mu
\]

\( \square \)

**Theorem 2** The Elementary Renewal Theorem. \( \frac{E[N(t)]}{t} \to \frac{1}{\mu} \) as \( t \to \infty \).

**Theorem 3** Blackwell’s Renewal Theorem.

(i) if \( F \) is not lattice, then \( m(t+a) - m(t) \to \frac{a}{\mu} \) as \( t \to \infty \) for all \( a \geq 0 \).

(ii) if \( F \) is lattice with period \( d \), then \( E[\text{the number of renewals at } nd] \to \frac{d}{\mu} \) as \( n \to \infty \).

**Notation.** A nonnegative random variable \( X \) is said to be lattice if \( \exists d \geq 0 \) such that \( \sum_{n=0}^{\infty} \text{Prob}\{X = nd\} = 1 \). In other words, \( X \) is lattice if and only if it only assumes integer multiples of \( d \).
**Definition 1** The period of $X$ is the largest $d$ having the above property.

We say the distribution function $F$ of $X$ is lattice if $X$ is lattice.

And now the idea behind the proof of Blackwell’s Theorem.

**Statement.** (A) If $F$ is not lattice, then the expected number of events in a length $b$ interval approaches $\frac{b}{\mu}$. Divide this into two parts.

(1) Show $\lim_{t \to \infty} [m(t + b) - m(t)] \equiv g(b)$ exists.
(2) Using (1) get

$$g(a + b) = \lim_{t \to \infty} [m(t + a + b) - m(t)]$$
$$= \lim_{t \to \infty} [m(t + a + b) - m(t + a) + m(t + a) - m(t)]$$
$$= g(b) + g(a)$$

Now the only increasing solution of $g(a + b) = g(a) + g(b)$ is $g(a) = ca$ for $a > 0$ and a constant $c$ – linearity.

(3) Now use the Elementary Renewal Theorem to prove that $C = \frac{1}{\mu}$, let

$$x_1 = m(1) - m(0)$$
$$x_2 = m(2) - m(1)$$
$$\vdots$$
$$x_n = m(n) - m(n - 1)$$

Then $\lim_{n \to \infty} X_n = C$ by definition and using $b = 1$. So

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} X_j = C$$

and so

$$\lim_{n \to \infty} \frac{m(n)}{n} = C$$
and then by the Elementary Renewal Theorem,

\[ C = \frac{1}{\mu} \quad (7.29) \]

Let \( N_j(t) \) equal the number of transitions into \( j \) by step \( t \). Let \( j \) be recurrent and \( X_0 = j \). Then \( \{N_j(t), t \geq 0\} \) is a renewal process (Poisson-type process) with inter-arrival distribution \( f_{jj}^n \) for \( n \geq 1 \).

**Proposition 6** If state \( j \) is transient then \( \sum_{n=1}^{\infty} P_{ij}^n < \infty \) for all \( i \).

**Corollary 1** If \( j \) is transient, then \( P_{ij}^n \to 0 \) as \( n \to \infty \).

**Definition 2** \( M_{jj} \) equals the number of transitions needed to return to state \( j \), i.e.,

\[ \mu_{jj} = \begin{cases} \infty & \text{if } j \text{ is transient} \\ \sum_{n=1}^{\infty} n f_{jj}^n & \text{if } j \text{ is recurrent} \end{cases} \quad (7.30) \]

**Definition 3** For \( j \) recurrent, we say it is positive recurrent if \( \mu_{jj} < \infty \) and null recurrent if \( \mu_{jj} = \infty \).

**Limit theorems.** If \( i \leftrightarrow j \) then

(i) \( P\{\lim_{t \to \infty} \frac{N_j(t)}{t} = \frac{1}{\mu_{jj}} | X_0 = i\} = 1 \).

(ii) \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{p_k}{n} = \frac{1}{\mu_{jj}} \).

(iii) If \( j \) is aperiodic, then \( \lim_{n \to \infty} P_{ij}^n = \frac{1}{\mu_{jj}} \).

(iv) If \( j \) has period \( d \) then \( \lim_{n \to \infty} P_{jj}^{nd} = \frac{d}{\mu_{jj}} \).

The proofs follow from the Renewal theorems and Poisson theories.