1. Probability – Lecture 1 – August 27 2002

1.1 Basic Notions

Let the letter $S$ represent a sample space. An event $U$ is an element of the sample space: $U \subseteq S$. So an event is a collection of certain outcomes.

1.2 Axioms of Probability

A probability space is described as $(S, P)$ for a sample space $S$, where:

(a) $P(U) \in [0, 1]$

(b) $P(S) = 1$

(c) $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ for any collection of mutually exclusive events $A_1, A_2, ... A_n$.

1.3 Idea of Random Variables

Example. Let $X$ be the outcome of a throw of a die. Let’s assume that $X$ takes on real values, that is, $X \in \mathbb{R}$. Furthermore assume the distribution function $F$ of the random variable $X$ is

$$F(X) = P\{X \leq x\} \quad (1.1)$$

where the event $U = \{X \leq x\}$.

For a discrete random variable, the analog of $F$ is the probability mass function $p(x)$, where

$$p(x) = P(X = x) \quad (1.2)$$

$$\sum_{i=1}^{n} p(x_i) = 1 \quad (1.3)$$

For a continuous random variable $X$, there is a non-negative $f(x)$ defined on
\[ P(X \in A) = \int_A f(x) \, dx \quad (1.4) \]

This \( f \) is called the **probability density function**.

**Proposition 1.**

\[
F(a) = P\{X \leq a\} = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x) \, dx \quad (1.5, 1.6, 1.7)
\]

Equivalently,

\[
\frac{d}{da} F(a) = f(a) \quad (1.8)
\]

### 1.4 Expectation / Averages

Define the expectation value \( E \) of a variable \( X \), for a discrete variable \( X \), to be

\[
E[X] = \sum_i x_i P\{X = X_i\} \quad (1.9)
\]

and for a continuous variable \( X \) to be

\[
E[X] = \int_{-\infty}^{\infty} xf(x) \, dx \quad (1.10)
\]

### 1.5 Application

**Proposition 2.** Let \( X \) be a discrete random variable having the probability mass function \( p(x) \). Then the expected value of the random variable \( Y = g(X) \) is given by

\[
E[g(X)] = \sum_i g(x_i)p(x_i) \quad (1.11)
\]
Proposition 3. In the continuous case,

\[ E\[g(X)\] = \int_{-\infty}^{\infty} g(x)f(x)dx \] \hspace{1cm} (1.12)

1.6 Linearity of \(E\)

Proposition 4. \(E[aX + b] = aE[X] + b\) for constants \(a, b\).

Proof. In the discrete case:

\[
E[aX + b] = \sum_i (ax_i + b)p(x_i) \tag{1.13}
\]

\[
= a \sum_i x_ip(x_i) + b \sum_i p(x_i) \tag{1.14}
\]

\[
= aE[X] + b \tag{1.15}
\]

In the continuous case:

\[
E[aX + b] = \int_{-\infty}^{\infty} (ax + b)f(x)dx \tag{1.16}
\]

\[
= a \int_{-\infty}^{\infty} xf(x)dx + b \int_{-\infty}^{\infty} f(x)dx \tag{1.17}
\]

\[
= aE[X] + b \tag{1.18}
\]

1.7 Variance

Define the variance of a variable \(X\) by

\[
Var(X) = E[(X - \mu)^2] \text{ where } \mu = E[X] \tag{1.19}
\]

Proposition 5. \(Var(X) = E[X^2] - \mu^2\).

Proof.

\[
Var(X) = E[(X - \mu)^2] \tag{1.21}
\]

\[
= E[X^2 - 2\mu X + \mu^2] \tag{1.22}
\]

\[
= E[X^2] - 2\mu E[X] + \mu^2 \tag{1.23}
\]
\[ E[X^2] - \mu^2 \] (1.24)

**Corollary.** \( \text{Var}(X) = E[X^2] - (E[X])^2 \)

**Proposition 6.** \( \text{Var}(aX + b) = a^2 \text{Var}(X) \)