2.1 Introduction to Classical Mechanics

2.2 N-Body Problems

Main ref: H. Goldstein and H. Pollard books [?]

\[ V(\vec{q}) = - \sum_{j \neq k} G \frac{m_j m_k}{r_{jk}} \]

\[ m_k \vec{r}_k = G \sum_{j=1, j \neq k}^N \frac{m_j m_k}{r_{jk}^2} \frac{\vec{r}_j - \vec{r}_k}{r_{jk}}, k = 1, ..., n \]  

(2.1)

This is an expression of Newton’s second law and his theory of gravitation.

Here

\[ r_{jk} \equiv |\vec{r}_j - \vec{r}_k| = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2} \]

It is not hard to show that

\[ \sum_{k=1}^N m_k \vec{r}_k = 0 \]

since the double sum

\[ G \sum_{j=1}^N \sum_{k=1, j \neq k}^N \frac{m_j m_k}{r_{jk}^2} \frac{\vec{r}_j - \vec{r}_k}{r_{jk}} = 0 \]

Let

\[ M_T = \sum_{j=1}^N m_j \]

be the total mass in the system. Let

\[ \vec{r}_c = \frac{1}{M} \sum_{j=1}^N m_j \vec{r}_j \]
be the center of mass. By the above lines,

\[ \vec{r}_c^\beta = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j^\beta = 0 \]

In other words the center of mass of an isolated system of particles moves in a straight line (or is stationary). This is clearly a generalization of Newton’s First Law of Motion to the case of a system of particles which is not under the influence of external forces. Stated differently, it is an expression of the principle of conservation of linear moment.

Like in the example on Kepler’s problem, we can transform to a coordinate system moving with the center of mass \( \vec{r}_c^\beta \), i.e.,

\[ \vec{r}_k \rightarrow \vec{r}_k - \vec{r}_c \equiv \vec{r}_k \]

or equivalently, we assume that the center of mass is fixed at the origin of the frame of reference, i.e.,

\[ \sum_{j=1}^{N} m_j \vec{r}_j = 0, \]

\[ \sum_{j=1}^{N} m_j \vec{r}_j = 0 \]

Then there are six constants of motion related to the conservation of linear momenta.

### 2.3 The Lagrange-Jacobi Identity

Let the moment of inertia be defined by

\[ I = \frac{1}{2} \sum_{j=1}^{N} m_j |\vec{r}_j^\beta|^2 \]

Computing:

\[ I'' = \sum_{j=1}^{N} m_j \left\{ (\vec{r}_j^\beta \cdot \vec{r}_j^\beta) + \vec{r}_j^\beta \cdot \vec{r}_j^\beta \right\} \]
2.4. THE VIRIAL THEOREM (AGAIN)

and using 2.1,

\[
I'' = \sum_{j=1}^{N} m_j v_j^2 + \sum_{j,k,j\neq k} G^{m_j m_k}_{r_{jk}^2} |\vec{r}_j^2 \cdot \vec{r}_k^2 - |\vec{r}_k^2|^2| \\
= 2KE + \frac{1}{2} \sum_{k \neq j} G^{m_j m_k}_{r_{jk}^3} |\vec{r}_j^2|^2 - |\vec{r}_k^2|^2 - r_{jk}^2
\]

which implies

\[
I'' - 2KE = \frac{1}{2} \sum_{k \neq j} G^{m_j m_k}_{r_{jk}^3} r_j^2 - \frac{1}{2} \sum_{k \neq j} G^{m_j m_k}_{r_{jk}^3} r_k^2 - \frac{1}{2} \sum_{k \neq j} G^{m_j m_k}_{r_{jk}^3}
\]

that is,

\[
I'' = 2KE + V(\vec{q}^2)
\]

Since \( E \equiv KE + V = h \), we have:

\[
I'' = KE + H = 2h - V
\]

2.4 The Virial Theorem (Again)

Suppose the evolution of a system of \( n \) particles is such that the potential energy \( V(\vec{q}^2) \) remains bounded for all time. If \( I \) and \( KE \) also remain bounded for all time \( t > 0 \), then the following holds: (i) \( KE \geq \lim_{t \to \infty} \frac{1}{t} \int_0^t KE(t) dt \) exists, (ii) \( 0 < V \geq \lim_{t \to \infty} \frac{1}{t} \int_0^t V(\vec{q}(t)) dt \) exists, and (iii) \( 2 KE + V = 2h - V \)

2.4.1

2.5 Canonical Transformations

Let old canonical variables be denoted by \((\vec{q}, \vec{p})\) where \( \vec{q} = (q_1, q_2, ..., q_N) \) and \( \vec{p} = (p_1, p_2, ..., p_N) \). New canonical variables will be denoted by \((\vec{Q}, \vec{P})\).
The relationship between these two sets of variables will henceforth be denoted by:

\[(\vec{q}, \vec{p}) = [F_q(\vec{Q}, \vec{P}), F_p(\vec{Q}, \vec{P})] \quad (2.2)\]

We say that \(F\) is a canonical transformation if the Jacobian matrix of \(F\), i.e.:

\[M \equiv \begin{bmatrix} \frac{\partial F_q}{\partial \vec{Q}} & \frac{\partial F_p}{\partial \vec{Q}} \\ \frac{\partial F_q}{\partial \vec{P}} & \frac{\partial F_p}{\partial \vec{P}} \end{bmatrix} \quad (2.3)\]

is a \textit{symplectic} matrix, where a \(2N \times 2N\) matrix \(M\) is symplectic if:

\[M^T J M = J\]

and

\[J \equiv \begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix}\]

The practical demonstration of symplecticity is best carried out in blocks as follows:

Let us consider the transformation

\[
\begin{align*}
q_1 &= Q_1 \cos Q_2 \\
q_2 &= Q_1 \sin Q_2 \\
p_1 &= P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2 \\
p_2 &= P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2
\end{align*}
\]

The associated Jacobian matrix is given by

\[M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\]

where
2.6. PRESERVATION OF HAMILTONIAN STRUCTURE UNDER CANONICAL TRANSFORMATIONS

\[
A = \begin{bmatrix}
\cos Q_2 & -Q_1 \sin Q_2 \\
\sin Q_2 & Q_1 \cos Q_2 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
\frac{P_2}{Q_1} \sin Q_2 & -P_1 \sin Q_2 - \frac{P_2}{Q_1} \cos Q_2 \\
-\frac{P_1}{Q_1} \cos Q_2 & P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2 \\
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
\cos Q_2 & -\frac{\sin Q_2}{Q_1} \\
\sin Q_1 & \frac{\cos Q_2}{Q_1} \\
\end{bmatrix}
\]

Using the fact that

\[
\begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix}^T = \begin{bmatrix}
A^T & C^T \\
B^T & D^T \\
\end{bmatrix}
\]

we check directly that \(M^T J M = J:\)

\[
\begin{bmatrix}
A^T & C^T \\
B^T & D^T \\
\end{bmatrix} \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0 \\
\end{bmatrix} \begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix} = \begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\]

2.6 Preservation of Hamiltonian Structure under Canonical Transformations

The Hamilton's equations in terms of the original variables are:

\[
\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \ldots, n
\]

Under the canonical transformation to new canonical variables \((\mathbf{Q}, \mathbf{P})\), it will be shown next that the equations of motion retain Hamilton's form.

The derivatives \(\dot{p}_k\) and \(\dot{q}_k\) are given by

\[
\dot{p}_k = \sum_{j=1}^{n} \left[ \frac{\partial p_k}{\partial P_j} \dot{P}_j + \frac{\partial p_k}{\partial Q_j} \dot{Q}_j \right]
\]

\[
\dot{q}_k = \sum_{j=1}^{n} \left[ \frac{\partial q_k}{\partial P_j} \dot{P}_j + \frac{\partial q_k}{\partial Q_j} \dot{Q}_j \right]
\]
according to the chain rule. In terms of the Jacobian $M$ in ?? we get

$$
\left( \frac{\vec{q}}{\vec{p}} \right) = M \left( \frac{\vec{Q}}{\vec{P}} \right)
$$

Since the coefficients

$$\left[ \frac{\partial q_k}{\partial Q_j} \right] = \frac{\partial F_{q}}{\partial Q}
$$

By exercise 3, $M^{-1} = -JM^T J$ exists; thus,

$$
\left( \frac{\vec{Q}}{\vec{P}} \right) = M^{-1} \left( \frac{\vec{q}}{\vec{p}} \right) = -JM^T J \left( \frac{\vec{q}}{\vec{p}} \right) = -JM^T \left( \frac{\vec{p}}{-\vec{q}} \right)
$$

Next, left-multiply on both sides by $J$ to get

$$
\left( \frac{\vec{P}}{-\vec{Q}} \right) = M^T \left( \frac{\vec{p}}{-\vec{q}} \right) = M^T \left( -\frac{\partial H}{\partial \frac{\partial q}{\partial p}} \right)
$$

since $J^2 = -J_{2N}$. By ??, we get

$$
\left( \frac{\vec{P}}{-\vec{Q}} \right) = \left( \begin{array}{cc} A^T & C^T \\ B^T & D^T \end{array} \right) \left( \begin{array}{c} -\frac{\partial H}{\partial \frac{\partial q}{\partial p}} \\ \frac{\partial F_{q}}{\partial \frac{\partial q}{\partial p}} \end{array} \right) \left( \begin{array}{c} -\frac{\partial H}{\partial \frac{\partial q}{\partial p}} \\ \frac{\partial F_{q}}{\partial \frac{\partial q}{\partial p}} \end{array} \right) \left( \begin{array}{c} -\frac{\partial H}{\partial \frac{\partial q}{\partial p}} \\ \frac{\partial F_{q}}{\partial \frac{\partial q}{\partial p}} \end{array} \right)
$$

which implies

$$
\dot{P}_k = -\sum_{j=1}^{N} \left[ \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial Q_k} + \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial Q_k} \right] = -\frac{\partial H}{\partial Q_k}
$$

and

$$
\dot{Q}_k = -\sum_{j=1}^{N} \left[ \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial P_k} + \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial P_k} \right] = \frac{\partial H}{\partial P_k}
$$
2.7 Applications of Canonical Transformations

We will return to an example discussed in Lecture 1, on Kepler's problem. Recall that we changed coordinates from \((q_1, q_2) \rightarrow (r, \theta)\) and \((p_1, p_2) \rightarrow (p_r, p_\theta)\) where:

\[(q_1, q_2) = re^{\theta}\]

and \(Q_1 = r, Q_2 = \theta, p_i = \dot{q}_i, i = 1, 2;\) and \(P_1 = \dot{r}\) and \(P_2 = r^2 \dot{\theta}\).

Let us reconstruct the map \(F\) that takes \((\vec{Q}, \vec{P})\) to \((\vec{q}, \vec{p})\) in the above example:

\[
\begin{align*}
q_1 &= r \cos \theta = Q_1 \cos Q_2 \\
q_2 &= r \sin \theta = Q_2 \sin Q_2
\end{align*}
\]

\[
\begin{align*}
p_1 &= \dot{r} \cos \theta - r \dot{\theta} \sin \theta \\
    &= P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2 \\
p_2 &= \dot{r} \sin \theta + r \dot{\theta} \cos \theta \\
    &= P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2
\end{align*}
\]

This is exactly the canonical transformation discussed earlier this lecture.

2.8 Exercise Set 1

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Exercise 1. Show that the transformation

\[p_j = Q_j, q_j = -P_j, j = 1, 2, \ldots, n\]

is canonical.

Exercise 2. Show that if a \(2N \times 2N\) matrix \(M\) is symplectic, so is \(M^{-1}\).

Exercise 3. Show that if \(M\) is symplectic then \(M^{-1} = -JM^TJ\).
Bibliography