Monotonicity of Social Opinion Dynamics on Large Networks

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Abstract

Motivated by the research on social opinion dynamics over large networks, a general framework for verifying the Monotonicity Property of multi-agent dynamics is introduced. This allows a derivation of sociologically meaningful sufficient conditions for monotonicity that are tailor-made for social opinion dynamics, which typically have high non-linearity. A key part of this framework is the definition of a partial order relation that is suitable for a large class of social opinion dynamics such as the generalized Naming games. Comparisons are made to previous techniques to verify monotonicity which fails on social opinion dynamics due to the high non-linearity of these systems. Using the results obtained, we extend many of the consequences of monotonicity to this class of social dynamics, including several corollaries on their asymptotic behavior, such as global convergence to consensus and tipping points of a minority fraction of zealots or leaders.

1 Introduction

Monotonicity is an appealing property for dynamical systems, as it provides a convenient method for analyzing the asymptotic behaviors such as convergence and stability of the system. Previous works on monotonicity include [8, 9] which are mainly used in applications to population genetics, biochemistry [22] and mathematical ecology [15]. However, when it comes to the case of social opinion dynamics, the direct application of previous monotonicity results, fail because the corresponding systems are highly non-linear[6],[19], [1], making it difficult to check their monotonicity by the Jacobian directly. Our main aim in this paper is to extend the applications of monotonicity to a large class of social opinion dynamics by providing an easy-to-verify criterion of monotonicity for this type of systems.

We introduce a mathematical framework comprehensive enough to represent many of the multi-agent social opinion dynamics in recent works such as the generalized Naming Games (NG) [1, 6, 10, 5] and the Voter models used in modeling of ad-hoc wireless networks, agreement on a small list of tags in the WWW and social opinion dynamics in forums [3] and elections. In this paper, the social opinion dynamics on networks of interacting agents allow a finite, but possibly large, number of opinion or information types [16], [17], [11]. Our aim is to show that the asymptotic behavior of a proper subset of these opinion dynamics on very large networks (where the effects of demographic noise can be safely neglected), can be described rigorously as a system of nonlinear coupled ODEs with the Monotonicity Property (MP) in phase space consisting of the coarse-grained population fractions of each opinion or information type.

The nonempty complement of the monotone social opinion dynamics includes two main types, namely, (1) the opinion dynamics where the associated random walk models are Martingales, that is, the deterministic drifts are zero everywhere in phase space; for instance, all versions of the Voter model [3] are diffusion - driven [2], [3], and (2) opinion dynamics which have nonzero drift almost everywhere in phase space but whose nonlinear ODEs are not monotone; we will give such a counterexample below.

In the mathematical framework given in detail below, the transmitted information or message is restricted to the binary symbols A, B. There are many ways to see why the binary-message case represents the end-game in the final stages of opinion dynamics based on more than two symbols. Concrete evidence of this has been discussed in [4, 3] where expected times to multi-consensus for a symmetric multi-opinions Voter model was calculated in terms of the bottleneck times when exactly two opinions remain viable in the game [3].

In the process of working out the mathematical and sociological consequences [11] [18] of monotonicity [8], we will first introduce a new way to construct a proper partial order that are explicitly designed for opinion dynamics. Along the way, we will also prove nontrivial extensions of the original monotonicity theorems in [8, 9]. Finally we will apply our main theorems on several examples and discuss the consequences for network science and mathematical sociology.

2 Framework for coarse-grained social opinion dynamics

Consider a social network consisting of N agents, each of which is dynamically assigned with an opinion s_i taking value from the opinion state space

$\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_K\}$. The (micro)state of the system is fully described by the vector of current opinions $\vec{S} = (s_1, ..., s_i, ..., s_N)$. In each time step, a speaker and a listener are randomly selected. The speaker sends a message containing 1 bit of information to the listener, and the listener changes its opinion state according to the message. In this context, the message can be "A" or "B". The probability for the selected speaker in state γ_k sending a message "A" is α_k. Let $\vec{\alpha} = (\alpha_1, ..., \alpha_K)^T$ ($\alpha_k \in [0, 1]$). When listener receives a message A (resp. B), the transition matrix of the listener's state is G_A (resp. G_B), the $i - j$th entry of which is $P(s_i \rightarrow s_j|A)$ for G_A. Both G_A and G_B are constant matrices and the dynamics of the network is fully governed by $\vec{\alpha}$, G_A and G_B.

A natural macrostate representation of this system is given by $\vec{n} = (n_1, ..., n_k, ..., n_K)^T$ where n_k is the fraction of nodes in spin state γ_k ($k = 1, ..., K$). The space of all possible macrostates is denoted by M, which is a simplex:

$$\begin{cases} \sum_{i=1}^{d} n_i = 1 \\ n_i \geq 0 \quad (i = 1, ..., d) \end{cases} \tag{1}$$

We refer to the vertices of M as pure macrostates in which all nodes of the system stay in the same spin state. For later use, let $\sigma : \Gamma \rightarrow M$ map a spin state to its corresponding pure macrostate. We rewrite the macrostate as a linear combination of pure macrostates.

$$\vec{n} = (n_1, ..., n_k, ..., n_K) = \sum_{k=1}^{K} n_k \sigma(\gamma_k)$$

On a complete network where every pair of agents are connected by an edge, assume at time step t, the macrostate is $\vec{n}(t)$; then at time step $t + 1$, the expected change of macrostate is

$$E[\vec{n}(t+1) - \vec{n}(t)] = \frac{1}{N} \left[pG_A + (1-p)G_B - I\right] \vec{n}(t),$$

where p is the overall probability for a message to be "A", given by

$$p = \vec{\alpha}^T \vec{n}.$$

With standard time scaling $dt = 1/N$, we obtain the mean field equation for the evolution of the macrostate:

$$\frac{d\vec{n}}{dt} = f(\vec{n}) = Q(\vec{n})\vec{n}(t) = \left[pG_A + (1-p)G_B - I\right]\vec{n}(t).$$

Here f maps M to its tangent space TM. As p is a linear function of $\vec{n}(t)$, this system has quadratic nonlinearity. All the social opinion dynamics discussed in this paper are generically highly nonlinear. The classical technique to verify monotonicity, as in the study of biological systems[22], uses the natural partial order induced by the positive orthant of M. Using this on social opinion dynamics often leads to a Jacobian $Df(\vec{n})$ with off-diagonal elements that change signs which implies the failure of classical technique. Hence, the necessity for a more sophisticated type of partial order for social opinion dynamics, which facilitates the verification of monotonicity in the corresponding mean field dynamical systems.

3 Partial order of type C_+

We begin with $\{\Gamma, \prec\}$, the opinion state space ordered by a partial order relation "$\prec$", satisfying:

$$\begin{cases} \text{reflexivity} : \gamma \prec \gamma \\ \text{symmetry} : (\gamma \prec \gamma') \wedge (\gamma' \prec \gamma) \Rightarrow \gamma = \gamma' \\ \text{transitivity} : (\gamma_1 \prec \gamma_2) \wedge (\gamma_2 \prec \gamma_3) \Rightarrow \gamma_1 \prec \gamma_3 \end{cases} \tag{2}$$

In later sections we will discuss what further conditions this partial order should satisfy. We induce a partial order relation between the pure macrostates from $\{\Gamma, \prec\}$ through $\sigma : \Gamma \to M$:

$$\sigma(\gamma) \prec \sigma(\gamma') \Longleftrightarrow \gamma \prec \gamma'$$

We use the same notation for the partial orders in both spaces with no ambiguity. Next we extend the partial order over Γ through affine combination. We require

$$\vec{n_1} \prec \vec{n_1}', \vec{n_2} \prec \vec{n_2}' \Rightarrow L(\vec{n_1}, \vec{n_2}) \prec L(\vec{n_1}', \vec{n_2}').$$

Here $L(\vec{n_1}, \vec{n_2}) = a\vec{n_1} + b\vec{n_2}$ is an affine combination of $\vec{n_1}, \vec{n_2}$ which satisfies $0 \leq a, b \leq 1$ and $a + b = 1$ so that $L(\vec{n_1}, \vec{n_2}) \in M$ whenever $\vec{n_1}, \vec{n_2} \in M$. Now the extended partial order, $\vec{n} \prec \vec{n}'$ if and only if $\vec{n}' - \vec{n}$ can be represented as

$$\vec{n}' - \vec{n} = \sum_i \lambda_i(\sigma(\gamma_i') - \sigma(\gamma_i))$$

The above summation is over all possible ordered pure macrostate pairs $\sigma(\gamma_i) \prec \sigma(\gamma_i')$ and $\lambda_i \geq 0$ for all i. Note that some terms in the summation may be further decomposed into combination of other ordered pairs, so we consider the independent set

$$B = \{\sigma(\gamma_i') - \sigma(\gamma_i) | \gamma_i \prec \gamma_i', \text{ and } \nexists \gamma'' st.\ \gamma_i \prec \gamma'' \prec \gamma_i'\}.$$

Let TM be the tangent space of macrostate space M. Then $C_+ \subset TM$ is the cone determined by the non-negative linear combination of set B, which is an analogue of the positive orthant in previous works on monotonicity [8, 9]. The definition of partial order of type $C+$ here can be described as

$$\vec{n} \prec \vec{n}' \Longleftrightarrow \vec{n}' - \vec{n} \in C_+ \tag{3}$$

If $|B|$, the cardinality of B, equals $dim(TM)$, the dimension of TM, the partial order of type $C+$ we described here can be considered as a standard type K partial order in Cartesian coordinate system ($x \prec y \Longleftrightarrow y - x \in \mathbb{R}^n_+$) after suitable linear transformation [8]. However, it is possible that $|B| \neq dim(TM)$ regarding some specific "$\prec$" on Γ (we will provide some examples later), therefore our partial order given by C_+ is a more general definition than the standard ones in the literature. We will show that this more general form is what we need for the applications in this paper to some well-known network games [4].

4 Criterion of monotonicity for opinion dynamics on complete networks

Denote ϕ_t as the semiflow which gives the solution of the ODE, $\vec{n}(t) = \phi_t(\vec{n}_0)$. A system is said to be order-preserving or monotone, if for $\forall \vec{n} \prec \vec{n}'$ and $\forall t > 0$, we have $\phi_t(\vec{n}) \prec \phi_t(\vec{n}')$.

We now give a condition for monotonicity, which is an analogue of the Kamke condition [8]. Assume $B = \{\vec{e}_1, ..., \vec{e}_{|B|}\}$. f is said to be ***type C*** if for each $k \in \{1, ..., |B|\}$, for $\forall \vec{n} \prec \vec{n}'$ such that $\vec{n}' - \vec{n} = \sum_{i \neq k} a_i \vec{e}_i$ ($a_i \geq 0$), there exists a representation of $f(\vec{n}') - f(\vec{n}) = \sum_{i=1}^{|B|} b_i \vec{e}_i$ in which $b_k \geq 0$. Note that when $|B|$ is greater than $dim(TM)$, the representation here may not be unique, but when $|B| < dim(TM)$, the representations may not exist regardless the sign of the coefficients.

Proposition 1(type C condition): The system $\frac{d\vec{n}}{dt} = f(\vec{n})$ is monotone if and only if it is type C.

The proof is straight forward by contradiction and very similar to that of the Kamke condition. If the type C condition does not hold, then by continuity of f, there exists an $\epsilon > 0$ such that $\phi_\epsilon(\vec{n}) \succ \phi_\epsilon(\vec{n}')$ which violates the monotone property. As the Kamke condition [21, 8] can be expressed in terms of partial derivatives, the type C condition has an expression in terms of directional derivatives.

Proposition 2: The system $\frac{d\vec{n}}{dt} = f(\vec{n})$ is monotone if and only if (A): for $\forall \vec{n} \in Int(M)$ and $\forall k \in \{1, ..., |B|\}$, there exists a representation $\frac{d}{d\epsilon} f(\vec{n} + \vec{e_k}) = \sum_{i=1}^{|B|} b_i \vec{e_i}$ s.t. for $\forall i \neq k$, $b_i \geq 0$.

Proof: For $\forall \vec{n} \prec \vec{n}'$ such that $\vec{n}' - \vec{n} = \sum_{i \neq k} a_i \vec{e_i}$ ($a_i \geq 0$) and the representation of $f(\vec{n}') - f(\vec{n}) = \sum_{i=1}^{|B|} b_i(\vec{n}) \vec{e_i}$ holds with the sign of b_i undecided,

$$
\begin{aligned}
f(\vec{n}') - f(\vec{n}) &= \int_0^1 \frac{d}{d\lambda} \vec{f}(\vec{n} + \lambda \sum_{i \neq k} a_i \vec{e_i}) d\lambda & (4) \\
&= \sum_{j \neq k} \int_0^1 a_i \frac{d}{d\epsilon} \vec{f}(\vec{n} + \lambda \sum_{i \neq k} a_i \vec{e_i} + \epsilon \vec{e_j}) d\lambda & (5)
\end{aligned}
$$

(a) (A)$\Rightarrow$type C: If conditon (A) holds, then $\frac{d}{d\epsilon} \vec{f}(\vec{n} + \lambda \sum_{i \neq k} a_i \vec{e_i} + \epsilon \vec{e_j}) = \sum_{i=1}^{|B|} b_i^{(j)}(\lambda) \vec{e_i}$ such that for $\forall i \neq j$, $b_i \geq 0$. We point-wisely replace the directional derivative by its linear representation, therefore

$$
f(\vec{n}') - f(\vec{n}) = \sum_{j \neq k} \int_0^1 a_i \sum_{i=1}^{|B|} b_i^{(j)}(\lambda) \vec{e_i} d\lambda. \qquad (6)
$$

Summing all the coefficients before $\vec{e_k}$ we get $\sum_{j \neq k} a_k \int_0^1 b_k^{(j)}(\lambda) d\lambda \geq 0$, so f is type C.

(b) Type C$\Rightarrow$(A): Suppose for some $\vec{n}_0 \in Int(M)$, $\frac{d}{d\epsilon} f(\vec{n}_0 + \epsilon \vec{e_j}) = \sum_{i=1}^{|B|} b_i \vec{e_i}$, $b_k < 0$ ($k \neq j$). By continuity of f, there exists a small enough $\delta_0 > 0$ such that $b_k < 0$ also holds for $\vec{n} = \vec{n}_0 + \delta \vec{e_j}$ when $0 < \delta \leq \delta_0$. Applying Eq. (6) $f(\vec{n}') - f(\vec{n}) = \delta_0 \sum_{i=1}^{|B|} \left(\int_0^1 b_i^{(j)}(\lambda) d\lambda \right) \vec{e_i}$. Since $b_k^{(j)}(\lambda) < 0$ for $0 \leq \lambda \leq 1$, the coefficient before $\vec{e_k}$ is $\delta_0 \sum_{i=1}^{|B|} \left(\int_0^1 b_k^{(j)}(\lambda) d\lambda \right) < 0$ contradicting the type C condition. QED

For the opinion dynamics on a complete network, we have $f(\vec{n}) = [pG_A + (1-p)G_B - I]\,\vec{n}(t)$, thus,

$$\begin{aligned}
\frac{d}{d\epsilon} f(\vec{n}+\epsilon\vec{e}_k) &= (pG_A + (1-p)G_B - I)\vec{e}_k + \left[\frac{dp(\vec{n}+\epsilon\vec{e}_k)}{d\epsilon}\frac{d}{dp}(pG_A + (1-p)G_B)\vec{n}\right] \\
&= pG_A\vec{e}_k + (1-p)G_B\vec{e}_k - \vec{e}_k + (\vec{\alpha}^T\vec{e}_k)(G_A - G_B)\vec{n}.
\end{aligned}$$

Now we derive a sufficient conditions for condition (A). According to Proposition 2, only b_i $(i \neq k)$ affects the monotonicity, so the third term in the last expression $(-\vec{e}_k)$ can be neglected. If we require the other three terms be positive in terms of $\prec$, we obtain the following theorem.

Theorem 1: An opinion dynamics on complete network governed by $\vec{\alpha}$, G_A, G_B is monotone if it satisfies

$$\begin{cases}
(a).\forall\gamma \in \Gamma,\ G_B\sigma(\gamma) \prec G_A\sigma(\gamma) \\
(b).\gamma \prec \gamma' \Rightarrow \vec{\alpha}^T\sigma(\gamma) < \vec{\alpha}^T\sigma(\gamma') \\
(c).\gamma \prec \gamma' \Rightarrow G_A\sigma(\gamma) \prec G_A\sigma(\gamma'), G_B\sigma(\gamma) \prec G_B\sigma(\gamma')
\end{cases} \tag{7}$$

This theorem is the main result of this paper. It gives convenient sufficient conditions on the essential components of the opinion dynamics in order for it to be monotone. The conditions in this theorem have sociological interpretations: Condition (a) fixes the preferred (greater) one between A and B to orient the partial order, that is, the listener will switch to a greater state according to this partial order when receiving A than when receiving B. Condition (b) says the speaker in a greater state has more probability to send a message A. One can switch the roles of A, B in the conditions (a), (b). Condition (c) says G_A and G_B preserve the partial order. In other words, we can now determine directly whether an opinion dynamics is monotone according to the properties of its three governing elements $\vec{\alpha}, G_A, G_B$. For particular applications of these sufficient conditions to social networks, we can imbue the partial order with a moral value system or a utility function [11]

5 Examples of binary-messaging system on complete graph

5.1 Binary Listener-only Naming Game (LO-NG) with and without committed agents

In this case [1], [6], the opinion types/states $\Gamma = \{A, AB, B\}$, and a macrostate is given by the corresponding populations $\vec{n} = (n_A, n_{AB}, n_B)^T$. Here, the governing elements are given by the following vector which fixes the probabilities of sending the symbol A when in the associated opinion states and a pair of transition matrices, which define the transition probabilities upon receiving a A (resp. B) symbol:

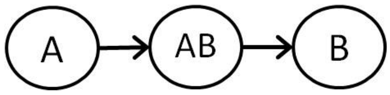
$$\vec{\alpha} = (1, 1/2, 0)^T,$$

$$G_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, G_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Thus, the governing elements of social opinion dynamics have two natural parts, namely, (I) the probabilities for sending a symbol, and (II) the transition probabilities to the next opinion state on receiving a symbol.

After choosing the opinion A to be "morally superior" say, and applying the implicit order in $\vec{\alpha}$, the suitable partial order over Γ that satisfies condition (b) in Theorem 1, is $B \prec AB \prec A$. Therefore, the independent subset of pairs, identified in the general mathematical framework given above, is $B = \{\sigma(A) - \sigma(AB), \sigma(AB) - \sigma(B)\} = \{(1, -1, 0), (0, 1, -1)\}$, where the negative ones in the last line denote the second element / term of the differences in the independent set B. In other words, the last set of 3-vectors are incidence vectors for the differences in B viewed as directed edges. Thus, this partial order can be represented by a directed graph, in which Γ gives the vertices and B gives the directed links, as in the figure below.

Figure 1: Partial order of NG



Next we check the conditions (a) and (c) in Theorem 1. In Table 1, comparing two rows, we verify (a) $G_B(\sigma(X)) \leq G_A(\sigma(X))$ $(X \in \{A, AB, B\})$; comparing three columns, we verify (c) $G_X(\sigma(B)) \prec G_X(\sigma(AB)) \prec G_X(\sigma(A))$ $(X \in \{A, B\})$. As a result, the monotonicity of NG on the complete graph without committed agents is verified.

Table 1: G_A, G_B of NG

	A	AB	B
$G_A(\sigma(\cdot))$	$\sigma(A)$	$\sigma(A)$	$\sigma(AB)$
$G_B(\sigma(\cdot))$	$\sigma(AB)$	$\sigma(B)$	$\sigma(B)$

In the case with committed agents both on A and B, $\Gamma = \{C_A, A, AB, B, C_B\}$ where C_A and C_B denote the additional opinion states, of being committed in A and B respectively. The populations macrostate is given by $\vec{n} = (n_{C_A}, n_A, n_{AB}, n_B, n_{C_B})^T$. The governing elements in this case are

$$\vec{\alpha_c} = (1, 1, 1/2, 0, 0)^T,$$

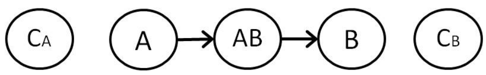
$$G_{A,c} = \begin{pmatrix} 1 & & \\ & G_A & \\ & & 1 \end{pmatrix}, G_{B,c} = \begin{pmatrix} 1 & & \\ & G_B & \\ & & 1 \end{pmatrix}.$$

The suitable partial order is shown in the directed graph below (just add two disconnected points to that of the non-committed or symmetric LO-NG case). Committed agents never change their opinion states, so $G_A(\sigma X) = G_B(\sigma X) = \sigma X$ $(X \in \{C_A, C_B\},)$ which therefore satisfy condition (a) in Theorem 1. Conditions (b) and (c) only affect the pairs ordered by the partial order. Since the committed case does not introduce any new ordered pairs w.r.t. the non-committed case, and $\vec{\alpha_c}, G_{A,c}, G_{B,c}$ restricted to the non-committed pure macrostates are exactly $\vec{\alpha}, G_A, G_B$, hence conditions (b) and (c) in Theorem 1 are satisfied.

From here, we can easily get the following corollary which has applications to the scenario in [13] and [14]:

Corollary: If an opinion dynamics without committed agents is monotone, adding committed agents into this social dynamical system preserve the monotonicity.

Figure 2: Partial order of NG with committed agents

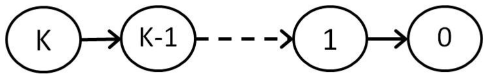


5.2 K-NG

In the one-parameter family of listener-only Naming Games, K-NG [10], where K presents the stubbornness of agents to full conversion from B to the A opinion, there are $K+1$ spin states $\Gamma = \{0, 1, ..., K\}$. As in the original

LO-NG case which corresponds to the value $K = 2$ in the $K - NG$ family of models, we firstly find a suitable partial order satisfying condition (b) in Theorem 1, that agrees with the given probability vector for sending the A symbol, $\vec{\alpha} = (0, 1/K, ..., i/K, ..., 1)^T$. This partial order is shown in the Fig. 3. Next we find the independent subset $B = \{\sigma(k+1) - \sigma(k) | k = 0, ..., K-1\}$.

Figure 3: Partial order of K-NG



Since the $K - NG$ family of NG models have the following property:

$$\sigma(max(k-1, 0)) \prec G_B(\sigma(k)) \prec \sigma(k) \prec G_A(\sigma(k)) \prec \sigma(max(k+1, K)),$$

conditions (a) and (c) in Theorem 1 follow easily, and the monotonicity of K-NG is verified.

5.3 Counter-example which is not monotone

Next we provide an explicit example of a binary-messaging system which is not monotone, thus establishing that the above definitions in our mathematical framework has non-vacuous complement. This example of binary-messaging network has the same opinion state space Γ and macrostate representation as the LO-NG model discussed previously. However, its governing elements (the probabilities for sending A and the spin transition probabilities upon receiving the symbols A or B) differ from the LO-NG in the matrix G_B:

$$\vec{\alpha} = (1, 1/2, 0)^T,$$

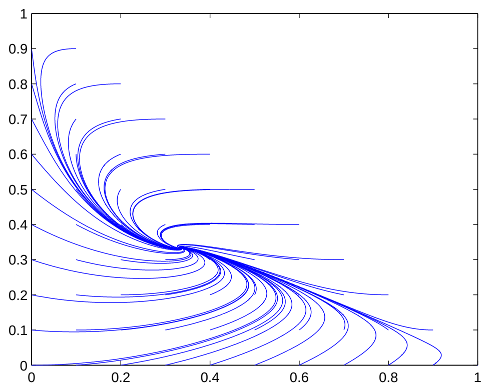
$$G_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, G_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We claim there does not exist a non-trivial partial order in this example, where by nontrivial we mean that the partial order has a nonempty independent subset $B \neq \emptyset$. The proof of this claim follows:

Proof: According to condition (b) in Theorem 1, there are at most three possible elements in B for this example, i.e. $B \subset \{\vec{e}_1 = \sigma(A) - \sigma(B), \vec{e}_2 = \sigma(AB) - \sigma(B), \vec{e}_3 = \sigma(A) - \sigma(AB)\}$. Since $G_A\vec{e}_1 = \vec{e}_3, G_A\vec{e}_2 = \vec{e}_3$, by (c) in Theorem 1, $\vec{e}_1 \in B \Rightarrow \vec{e}_3 \in B$ and $\vec{e}_2 \in B \Rightarrow \vec{e}_3 \in B$. If $B \neq \emptyset$, then $\vec{e}_3 \in B$. However, $G_B\vec{e}_3 = \sigma(B) - \sigma(AB)$, by (c) again, $\sigma(AB) \prec \sigma(B)$ which contradicts condition (b) in Theorem 1. QED

The solution trajectories of this system mapped into 2D space (n_A, n_B) is shown below. This figure gives us a concrete idea of one type of non-monotone opinion dynamics on two symbols.

Figure 4: Trajectories of counter example



6 Monotonicity on sparse random networks

Up to this point we have discussed social opinion dynamics on large complete networks which are the easiest ones for which we have mean field equations. However, the most interesting cases of social opinion dynamics evolve on large social networks which have the small-world [25] and/or scale-free properties [24]. As far as we know tractable mean field equations for sophisticated social opinion dynamics on these networks remain an open problem. Hence, an intermediate step in the mathematical modelling of sophisticated or cognitive opinion dynamics, is to derive the mean field equations for coarse-grained information-type population fractions on large random networks such as the Erdos-Renyi graphs.

Under a homogeneous pairwise mean field assumption, the coarse-grained dynamics of the binary Naming Game on a random sparse network with average degree $< k >$ is governed by the 6-dimensional ODEs [12] [19]:

$$\frac{d}{dt}\vec{l} = 2\left[\frac{1}{< k >}D + (\frac{< k > -1}{< k >})R\right]\vec{l}. \tag{8}$$

for macrostate $\vec{l} = [l_{A-A}, l_{A-B}, l_{A-AB}, l_{B-B}, l_{B-AB}, l_{AB-AB}]^T$ of link-types population fractions. For a general social system of the above type, we can obtain a similar ODE system using exactly the same approach as in [19]. The type of a link in this ODEs is ultimately given by the opinions or node-spins of its two ends $(\gamma_i - \gamma_j)$ regardless of the order but it is more convenient to work with link-based macrostates; a transformation between the link-based macrostate and the node-based macrostate $\vec{n}$ is given in [19]. Changes of $\vec{l}$ come from two parts: the ***direct change*** and the ***related change***. In each time step, a realized listener-speaker pair of agents and therefore a link or edge in the underlying random graph is selected. Then

Direct change is the change of the selected link and is given by $D\vec{l}$ where D is the probability transition matrix of the selected link - the (i, j) entry of D, D_{ij}, is the probability that a link of type j changes into type i in one step given that the selected link is of type j. D is a constant matrix given by $\vec{\alpha}$, G_A, G_B and the way of selecting listeners and speakers.

Consider a link type $\gamma_1 - \gamma_2$ and its corresponding pure macrostate $\sigma(\gamma_1 - \gamma_2)$ which, according to the general mathematical framework in this paper, are now the basic elements on which a partial order is defined. Then according to this theory and the above 6-dimensional ODEs (with effectively a 5-dim tangent space after reduction by one degrees of freedom),

$$\begin{aligned} D\sigma(\gamma_1 - \gamma_2) \quad &= \quad P(\gamma_1 \text{ is listener})\left[p_1\sigma(G_A\gamma_1 - \gamma_2) + (1-p_1)\sigma(G_B\gamma_1 - \gamma_2)\right] \\ &\quad + P(\gamma_2 \text{ is listener})\left[p_2\sigma(\gamma_1 - G_A\gamma_2) + (1-p_2)\sigma(\gamma_1 - G_B\gamma_2)\right], \end{aligned}$$

where $p_1 = \vec{\alpha}^T\sigma(\gamma_2)$, $p_2 = p_1 = \vec{\alpha}^T\sigma(\gamma_1)$. The notations $\sigma(G_A\gamma_1 - \gamma_2)$ is defined by

$$\sigma(G_A\gamma_1 - \gamma_2) = \sum_{i=1}^{K} P(G_A\sigma(\gamma_1) = \sigma(\gamma_i))\sigma(\gamma_i - \gamma_2)$$

The related links are those incident at the listener other than the selected link. Then the ***Related change*** is the change of the related links when the listener changes and is given by $(< k > -1)R(\vec{l})\vec{l}$, where $< k > -1$ is the expected number of related links. R is a probability transition matrix that varies according to the current macrostate. For Naming Game, R is given explicitly in [19]. For general opinion dynamics, R is given in detail later.

A natural partial order of link-based macrostates is induced from that of node-based macrostates: the link states

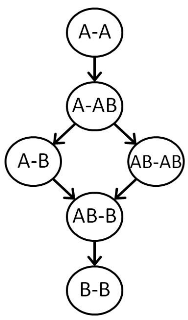
$$X - Y \prec X' - Y' \Longleftrightarrow X \prec X' and Y \prec Y'.$$

The independent set of ordered pairs B is thus

$$B = \{\sigma(X - Y') - \sigma(X - Y) | Y \prec Y'\}.$$

For example, the partial order and the set B for NG on a sparse random network is shown in the following figure. Note that it is an example in which $|B| = 6 \geq dim(TM) = 5$. The partial order in the link-based

Figure 5: Partial order of NG on random sparse networks



macrostate space is stronger than that in the node-based macrostate space, i.e. if two node-based macrostates are $\vec{n}, \vec{n}'$ and their respective linked based macrostates $\vec{l}, \vec{l'}$, then $\vec{l} \prec \vec{l'} \Rightarrow \vec{n} \prec \vec{n}'$ and the reverse does not hold.

In Section 4, we proved a sufficient condition for an opinion dynamics to be monotone on a complete network graph. We will show in the following theorem, that the opinion dynamics satisfying this condition will also be monotone on a random sparse network.

Theorem 2: If an opinion dynamics satisfies the conditions in Theorem 1, then it is monotone on a random sparse network.

Proof: Firstly, we show that given the conditions in (6), the direct change part $D\vec{l}$ satisfies condition (a). For each $\vec{e}_k = \sigma(X-Y') - \sigma(X-Y)$,

$$\begin{aligned} D\vec{e}_k &= D\sigma(X-Y') - D\sigma(X-Y) \\ &= P(X \text{ is listener})[\vec{\alpha}^T\sigma(Y)\left(\sigma(G_AX-Y') - \sigma(G_AX-Y)\right) \\ &\quad + \left(1-\vec{\alpha}^T\sigma(Y)\right)\left(\sigma(G_BX-Y') - \sigma(G_BX-Y)\right) \\ &\quad + \left(\vec{\alpha}^T\sigma(Y') - \vec{\alpha}^T\sigma(Y)\right)\left(\sigma(G_AX-Y') - \sigma(G_BX-Y')\right)] \\ &\quad + P(Y \text{ or } Y' \text{ is listener})[\vec{\alpha}^T\sigma(X)\left(\sigma(X-G_AY') - \sigma(X-G_AY)\right) \\ &\quad + \left(1-\vec{\alpha}^T\sigma(X)\right)\left(\sigma(X-G_BY') - \sigma(X-G_BY)\right)] \end{aligned}$$

By the definition of link-based partial order, $\sigma(G_AX-Y') - \sigma(G_AX-Y) \succ 0$, $\sigma(G_BX-Y') - \sigma(G_BX-Y) \succ 0$. According to condition (6), $\vec{\alpha}^T\sigma(Y') - \vec{\alpha}^T\sigma(Y) > 0$, $\sigma(X-G_AY') - \sigma(X-G_AY) \succ 0$, $\sigma(X-G_BY') - \sigma(X-G_BY) \succ 0$. Therefore $D\vec{l}$ preserve the order of l

For the related change part, $R\vec{l}$, we represent $\vec{l}$ as a suitably weighted symmetric adjacency matrix $M = M(\vec{l})$ labeled by the node-based spin types,

$$\begin{cases} M_{ii} = l_{\gamma_i-\gamma_i}, \\ M_{ij} = \frac{1}{2}l_{\gamma_i-\gamma_j}(i \neq j). \end{cases} \tag{9}$$

where the row sum and column sum of $M(\vec{l})$ are the node-based macrostate, $\vec{n}$. It is obvious that, $M(\vec{l})$ is a $1-1$ presentation for $\vec{l}$. $R(\vec{l})$ is given by the following

$$M(R(\vec{l})\vec{l'}) = W(\vec{l})M(\vec{l'})W(\vec{l})^T,$$

where $W(\vec{l})$ is the transition matrix of spin states, i.e. the entry of $W(\vec{l})$, W_{ij} is the probability that a node in spin state γ_j changes into γ_i given the link-based macrostate $\vec{l} = (l_1, ..., l_k, ..., l_K)$,

$$W_{ij} = \sum_{k=1}^{K} P(\gamma_j \to \gamma_i| \text{ link type } \ k) l_k.$$

Then we prove the following lemmas.

Lemma 1: There exists a unique decomposition of $M(\vec{l})$, $M(\vec{l}) = u + v$. $u \in U = \{u | u = u^T, \exists \text{column vector} \vec{m} \text{ s.t. } u = \vec{m} \otimes \vec{m}^T\}$, $\otimes$ is the kronecker product . $v \in V = \{v | v = v^T, v\vec{\mathbf{1}} = \vec{0}, \vec{\mathbf{1}}^T v = \vec{0}\}$, where $\vec{\mathbf{1}}$ is a column vector with all 1 entries.

Proof: Taking $\vec{m} = \vec{n}$ the node-based population fractions, $u = \vec{n} \otimes \vec{n}^T$ is symmetric by construction. The row sum and column sum of u are both $\vec{n}$, the same as $M(\vec{l})$. Therefore the row sum and column sum of v are 0. As $M(\vec{l})$ and u are symmetric, so is v.

Lemma 2: U and V are invariant space of the operator $R(\vec{l})$ for any macrostate $\vec{l}$.

Proof: $\forall u = \vec{m}^T \otimes \vec{m} \in U$, $W u W^T = (W\vec{m}) \otimes (W\vec{m})^T \in U$. $\forall v \in V$, $v\vec{\mathbf{1}} = \vec{0}$, since $W^T \vec{\mathbf{1}} = \vec{\mathbf{1}}$, $W v W^T \vec{\mathbf{1}} = \vec{\mathbf{1}}^T W v W^T = \vec{0}$, therefore $W v W^T \in V$.

According to Lemma 1 and 2, $WM(\vec{l})W^T$ restricted on U is just $(Q\vec{n}) \otimes (Q\vec{(n)})^T$, where Q is the transition matrix of the corresponding dynamics on complete network, $\vec{n} = M(\vec{l})\vec{\mathbf{1}}$ is the corresponding node-based macrostate. Besides, $WM(\vec{l})W^T$ restricted on V has zero effect on the dynamics of $\vec{n}$. So $WM(\vec{l})W^T$ preserve the partial order of $\vec{n}$. This completes the proof of theorem 2.

7 Consequences

In previous sections, we established sufficient conditions for social opinion dynamics to have the Monotonicity Property in Theorems 1 and 2. To aid the application of monotonicity to social opinion dynamics, we introduce one additional property: ***approximate from below (above)***.

Definition: x can be approximated from below (above), if there is a sequence $\{x_n\}$ satisfying $x_n \prec x_{n+1} \prec x$ ($x_n \succ x_{n+1} \succ x$) for $n \geq 1$ and $x_n \to x$ as $n \to \infty$.

With the partial order of type C_+ discussed in this paper, every point x in macrostate space can be approximated from below and above except for the two consensus state, since the approximating sequence is given by $x + \epsilon_n \vec{e_i}$ with $\vec{e_i} \succ 0$ and $\epsilon_n \to 0_-(0_+)$. Besides, the macrostate space for opinion dynamics is always a simplex, and therefore it is finite-dimensional, convex and compact.

Consequently using Theorems 1, 2 and this aditional property, many previously inapplicable results of monotone dynamical system in [20, 8, 9], such as the following three theorems, are now directly applicable to a large class of social opinion dynamics, including many cases reviewed in the recent literature [14],[16],[17],[5],[7]. In particular the resulting substantial reduction in complexity of the phase trajectories and the organization of the phase space into hyperbolic equilibria and the heteroclinic orbits that connect them have clear implications for mathematical sociology and network science [6],[19], [1]. In view of the properties above of social opinion dynamics, the following two theorems in [8] can now be applied to a class of problems that are relevant to mathematical sociology, such as the generalized Naming games. They imply that the global convergence of a monotone opinion dynamics is simply decided by the equilibria.

Theorem - Global asymptotic stability [8, 9]: *If a monotone opinion dynamics contains exactly one equilibrium e, then every initial macrostate converge to e.*

Theorem - Tipping point [6],[19], [1],[8, 9]: *If a monotone opinion dynamics has two equilibria* $x \prec y$*,* $[x, y]$ *denotes the set of all the points* z *that* $x \prec z \prec y$*, then one of the following holds:*
1) y *is stable, every point except* x *converges to* y*.*
2) x *is stable, every point except* y *converges to* x*.*
3) x, y *are both stable, and there exists another equilibrium* $z \in [x, y]$*,* $z \neq x, y$*.*

In these two theorems, the former one guarantees the global convergence of an isolated equilibrium without information on the stability of the equilibrium. The latter one is especially relevant to the "tipping point" phenomenon found in the binary Naming games [6],[19], [1]. It predicts all possible global structures of the binary NG dynamics from only their monotonicity property instead of detailed inter-agent rules. Therefore the previous results obtained for the binary NG [6],[19], [1], [5] can now be qualitatively generalized to any monotone binary-message systems with two consensus states. If the opinion dynamics contains more than two ordered stable equilibria, say $x \prec y \prec z$, then the tipping point conclusion can be applied on the domains (called attracting basins) $[x, y]$ and $[y, z]$ separately.

Furthermore, a theorem in [8] implies the useful corollary on the global convergence of monotone opinion dynamics:

Theorem - Convergence [8, 9] *For any given monotone opinion dynamics,* $M = \overline{Int(C)}$ *where* M *is the macrostate space and* C *is the set of points that will eventually converge (to some equilibria).*

In other words, $Int(\overline{C^c}) = \emptyset$, the set of points that do not converge to anywhere is a nowhere dense set. Considering that the opinion dynamics in the real world always contain some noise, it is impossible for the trajectory to stay inside a nowhere dense set even with infinitely small noise. Therefore a monotone opinion dynamics starting from any initial state will eventually go to a stable equilibrium state.

Taken together, the results in this paper are applicable to many of the convergence, coherence or synchrony questions that arise in mathematical sociology. However, the method and results in this paper are currently limited to complete networks or large networks with simple random properties like ER networks[23]. In future studies, it may be possible to extend the monotonicity properties to the mean field equations on more realistic social networks such as scale-free networks[24] and small-world networks[25].

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References

[1] J. Xie, S. Sreenivasan, G. Korniss, W. Zhang, C. Lim, and B. K. Szymanski, 2011, Social consensus through the influence of committed minorities, Phys. Rev. E 84, 011130.

[2] M Starnini, A. Baronchelli, and R. Pastor-Sartorras, 2012, Ordering dynamics of the multi-state voter model, J. Stat. Mech, P. 10027.

[3] C. Lim, Information sharing with strong neutrals- exact solutions for expected times to multi-consensus, submitted Phys Rev E May 2013.

[4] W. Zhang, 2013, Analytical approach for opinion dynamics on social networks. PhD thesis. RPI.

[5] A. Baronchelli, V. Loreto, and L. Steels, 2008, In-depth analysis of the Naming Game dynamics: the homogeneous mixing case, Int. J. Mod. Phys. C 19, 785.

[6] W. Zhang, C. Lim, S. Sreenivasan, J. Xie, B. K. Szymanski, and G. Korniss, 2011, Social influencing and associated random walk models: Asymptotic consensus times on the complete graph, Chaos 21, 025115.

[7] X. Castellano A. Baronchelli, and V. Loreto, 2009, Consensus and ordering in language dynamics, European Physical Journal B 71, 557-564.

[8] H. L. Smith, Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems, Mathematical survey and monographs Vol 41. 1995.

[9] H.L. Smith and H. Thieme, Quasi convergence for strongly order-preserving semiflows, SIAM J. Math. Anal. 21 1990.

[10] Y. Treitman, C. Lim, W. Zhang and A. Thompson, Naming Game with Greater Stubbornness and Unilateral Zealots, IEEE Conf Proc West Point NY May 2013.

[11] M. Granovetter. Threshold Models of Collective Behavior. *American Journal of Sociology*, 83, 6: 1420C1443 (1978).

[12] W. Zhang, C. Lim, G. Korniss, B. Szymanski, S. Sreenivasan and J. Xie: Tipping Points of Diehards in Social Consensus on Large Random Networks. Proc. 3rd Workshop on Complex Networks, CompleNet, Melbourne, FL, March 7-9, 2012.

[13] J. Xie, J. Emenheiser, M. Kirby, S. Sreenivasan, B. K. Szymanski, and G. Korniss, Evolution of opinions on social networks in the presence of competing committed groups, PLoS ONE 7(3):e33215. doi:10.1371/journal.pone.0033215, 2012.

[14] Gunjan Verma, Ananthram Swami, Kevin Chan, The Effect of Zealotry in the Naming Game Model of Opinion Dynamics, MILITARY COMMUNICATIONS CONFERENCE, 2012 - MILCOM 2012 p. 1-6, Orlando, Oct. 29 2012-Nov. 1 201, IEEE Computer Science Press

[15] Earn, D.J.D. and S. A. Levin, Global asymptotic coherence in discrete dynamical systems. Proceedings of the National Academy of Sciences, USA 103 (11): 3968-3971, 2006.

[16] Ann P. Kinzig , Paul R. Ehrlich , Lee J. Alston , Kenneth Arrow , Scott Barrett , Timothy G. Buchman , Gretchen C. Daily , Bruce Levin , Simon Levin , Michael Oppenheimer , Elinor Ostrom and Donald Saari, "Social Norms and Global Environmental Challenges: The Complex

Interaction of Behaviors, Values, and Policy", BioScience, 63(3):164-175. 2013. http://www.bioone.org/doi/full/10.1525/bio.2013.63.3.5

[17] Malgorzata Turalska, Bruce J. West and Paolo Grigolini, Role of committed minorities in times of crisis, SCIENTIFIC REPORTS $|3 : 1371|DOI : 10.1038/srep01371$

[18] Thomas Schelling£ Micromotives and Macrobehavior. Norton (1978).

[19] W. Zhang, C. Lim, B. Szymanski, Analytic Treatment of Tipping Points for Social Consensus in Large Random Networks, Phy. Rev. E 86 061134 (2012).

[20] M.W. Hirsch 1985. Systems of Differential Equations that are Competitive or Cooperative II: Convergence Almost Everywhere. SIAM J. Math. Anal., 16(3), 423-439.

[21] E. Kamke 1932. Zur Theorie der Systeme gewohnlicher Differentialgleichungen. II. Acta Math. 58, 57-85.

[22] Eduardo D. Sontag(2007). Monotone and near-monotone biochemical networks. Systems and Synthetic Biology. Vol 1, Issue 2, 59-87.

[23] P. Erdos, A. Renyi (1960). On the evolution of random graphs. Publ. Math. Inst. Hungar. Acad. Sci, 1960.

[24] R. Albert, A. Barabasi (2002). Statistical mechanics of complex networks. Reviews of Modern Physics 74: 47-97.

[25] DJ. Watts, SH. Strogatz (1998). Collective dynamics of 'small-world' networks. Nature 393 (6684): 440-442.