A Monte Carlo Algorithm for Free and Coaxial Ring Extremal States of the Vortex N-Body Problem on a Sphere

By
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Abstract

We show that the search for statistical equilibria at very low positive temperatures, using a Monte Carlo algorithm, can successfully locate dynamical equilibria of the N-vortex problem on a sphere. Numerical results are collected to show that for a wide range of particle numbers, this algorithm accurately and efficiently locates the ground state or lowest energy equilibrium. The extremal states found numerically are carefully compared with well-known exact configurations such as the regular polyhedra. Using an essential tool called the radial distribution function, we state a theorem that is useful for comparing N-vortex configurations which are related to one another by elements of the group $O(3)$. It is found that by constraining the system to equally spaced latitudinal rings of vortices the computational cost may be reduced by an order of magnitude. Many of the results reported here apply directly to other N-body problems on a sphere, such as the distribution of N charges on a sphere.

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1 Introduction

Extremal configurations in N-body problems, such as relative equilibria of point vortices on a sphere, are intrinsically interesting. They also have many applications in diverse fields including to the determination of optimal distributions of point charges on spherical surfaces [8] [15]. Such problems naturally have a variational formulation in terms of a energy functional or Hamiltonian. In the case of the model vortex problem considered here, we will take the Hamiltonian to be given [14] by

$$H(\vec{s}) = -\sum_{j<k} \Gamma_j \Gamma_k \log(1 - z_j \cdot z_k) \tag{1}$$

which applies equally well to the vortex on S^2 problem as to the Coulomb problem on a sphere. Another special quality of this problem is the significant role of symmetry in both its theoretical framework and its practical resolution [14] [5]. We give a theorem which states that the radial distribution function discussed herein is an ideal tool for classifying vortex configurations in terms of $O(3)$ symmetries.

With this preamble in mind, the first aim here is to formulate and prove the efficacy of a novel computational strategy for the solution of this variational problem. The numerical technique in mind is a statistical method known as the Metropolis Algorithm in Monte-Carlo simulations. At first sight, it may be surprising that a manifestly statistical method can be used to efficiently solve a deterministic variational problem. Upon further inspection and recollection of the fact that the Metropolis Algorithm is based on maximizing the Gibbs factor

$$P(\vec{s}) = \frac{1}{Z} \exp(-\beta H(\vec{s})), \tag{2}$$

the relevance (if not yet the efficacy) of the Monte-Carlo method surfaces. Nonetheless, it should be evident to those experienced in Monte-Carlo that we are proposing to use Monte-Carlo in an unorthodox way, that is, to find the dynamical equilibria of a class of N-body problems, when the traditional usage of MC is to compute statistical equilibria in a canonical ensemble.

Furthermore, the algorithm outlined here is not dependent upon the vortex gas Hamiltonian – it was studied in this problem because of the many results known analytically which allow the validation of this method, but may in principle be applied to any pairwise potential energy function with the same symmetry.

In the following sections (2, 2.3, 2.5), are shown the computation of low energy N-vortex configurations on a sphere.

An important tool from computational statistical mechanics will be introduced (section 2.6) to fingerprint the low energy states found by this method: the radial distribution function or reduced two particle distributions of these vortex states will be computed as natural by-products of the Monte-Carlo algorithm.

Also introduced is a means of building equilibrium configurations is the notion of constraining the system to a number of rings of vortices, with each vortex of uniform strength and the vortices along each ring uniformly spaced in longitude (section 2.5). This is shown to bring a great reduction in the computational time required for the system (section 2.4) without losing the excellent approximations of extremal states for the N-body problem.

2 The Monte Carlo Algorithm and the Radial Distribution Function (rdf)

The dynamical system explored in this paper is that of n point vortices on the surface of the sphere. These systems have been analytically explored and quite a few relative equilibria are known by Lim, Montaldi, and Roberts to exist [14] and have been classified not only to shape but also to dynamic stability. This paper examines only the shape, the patterns of those equilibria.

We make use of the Monte Carlo algorithm to generate relative equilibria of a point vortex system by searching for statistical equilibria of the associated vortex gas. The system examined here has the advantage of representation by a set of Hamiltonian equations (despite being a non-Newtonian problem) [9] [13] [14] [16], suggesting the method should extend well to traditional mechanical problems.

A distinct advantage to Monte Carlo methods in seeking relative equilibria is their ability to filter out oscillations in time around any given relative equilibrium: there will not be particles oscillating around an equilibrium because there is no time in the dynamic sense. Particles in a system go to a position and, apart from whatever fluctuations might be allowed by the Monte Carlo routine, do not move.

2.1 The Monte Carlo Sweep

Given a system of n point vortices, all of uniform strength (assumed to be 1), the potential energy will be

$$H(\vec{z}_1, \vec{z}_2, ..., \vec{z}_n) = -\sum_{j<k}^{n} \log |1 - \vec{z}_j \cdot \vec{z}_k| \tag{3}$$

for particles on the surface of the sphere (with $\vec{z}_j = (x_j, y_j, z_k)$ the coordinates of vortex j, and the constraint that $|\vec{z}_j| \equiv 1$ for all j) [9] [13] [16].

A Monte Carlo algorithm following the Metropolis rule is employed to find an equilibrium configuration for this system. A sweep is a series of n experiments – matching the number n of vortices – in each of which one vortex is selected (randomly, although traditional Monte Carlo methods allow the selection to be done in a preselected order). On each vortex a proposed displacement is tested: Consider moving the chosen vortex by a random amount – a rotation around a randomly selected axis by an angle uniformly distributed on $(0, \epsilon)$. The change in energy ΔH this rotation would cause is then calculated. Based on a preselected inverse temperature β, and a randomly chosen number r uniformly distributed on $(0, 1)$, the change is

$$\left\{ \begin{array}{ll} \text{accepted if} & r < \exp\ -\beta \Delta H \\ \text{rejected if} & r \geq \exp\ -\beta \Delta H \end{array} \right\}$$

This is the Metropolis Rule for Monte Carlo algorithms. (Note that if β is positive this results in any change which decreases energy being accepted. A negative inverse temperature β, while not theoretically objectionable for this problem, is uninteresting – it results in the vortices clustering together, rather than to spreading out over the physical domain.)

This process of consideration and acceptance or rejection is repeated for each of the n vortices in the sweep. The sweeps are repeated until either a statistical equilibrium is achieved or a preselected number of sweeps has been allowed to elapse. The measure of a statistical equilibrium is to examine the fluctuations in the radial distribution function – a chart of the pairwise angular separations [6] – and watch for the fluctuations to decline to a chosen minimum. In this paper waiting for a preselected number of sweeps to complete is used.

This process of approaching ground states through a Monte carlo algorithm does not require only a system of free particles. It can be applied to systems of particles in which further constraints are made – for example, requiring several vortices to be aligned on a little circle, and instead of moving one vortex at a time moving the entire circle as a rigid body. This opens the prospect of finding equilibriums for systems with many possible constraints and will be explored in detail in section 2.5.

2.2 Convergence Time

An approach to determining the length of a computation needed to find a ground state is plainly empirical, from observing the energy of the system and waiting for it to reach a nearly constant value. For small numbers of vortices many experiments can be run and the typically rapid convergence offers support to the notion that for reasonably good estimates only a few hundred sweeps are necessary.

In practice several hundred sweeps suffices, as it does for the illustrated examples of several numbers of vortices in figure 1, to settle to a configuration close to the ground state. (The energy per particle pair is illustrated as these numbers are of approximately the same order. Energy by itself grows as the square of the number of particles [2] and so would be difficult to plot all the above experiments on the same axes.)

One minor difficulty of the Monte Carlo approach is that while convergence to a configuration close to the ground state is rapid, there is difficulty in getting to a precise ground state. This is the unavoidable result of the randomness needed for this mechanism to work at all. If a vortex is 0.0001 radians away from the ideal position, the chance that a move of just that distance and just that direction will be considered is tiny, and even the chance of moves drawing closer to that point being proposed grows small rapidly.

The example of the energy after each sweep as in figure 1 highlights that the energy function is such that the random placement of vortices is not likely to produce a low-energy state, though it will be demonstrated that a vortex arrangement not being nearly uniform does not by itself require the system energy to be significantly higher than the ground state energy.

2.3 Free Configurations

Given a number n of vortices an initial configuration is found by placing them randomly on the surface of the unit sphere. From this initial configuration a large number of Monte Carlo sweeps are made, using an extremely high value

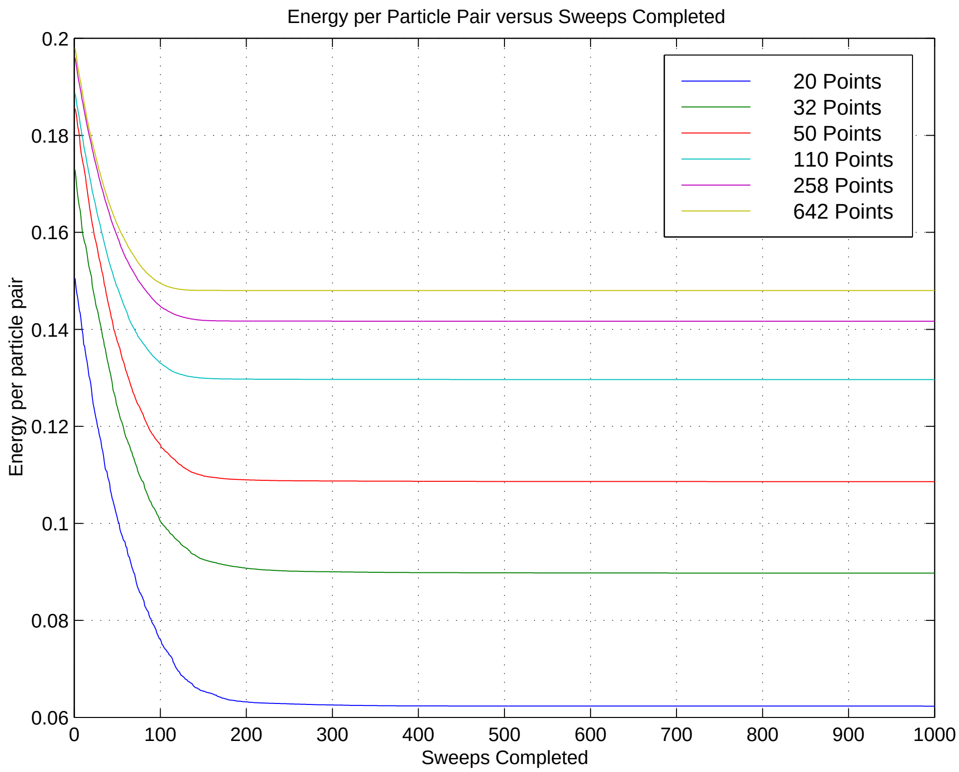


Figure 1: System energy (per pair of points) versus the number of sweeps completed for 20, 32, 50, 110, 258, and 642 points on the surface of the sphere. Convergene occurs within several hundred sweeps, as it appears to do for the range of N values examined in this paper.

for the inverse temperature β. This has the effect of allowing any proposed move of a vortex which would decrease the system energy, and allowing few or no moves which would increase the system energy – the goal is a nearly relentless decrease in the system energy [1]. This configuration must represent an equilibrium of the system – the Hamiltonian's global minimum must be at least a relative equilibrium – and so is the object of interest.

2.4 Reduction of Problem Order

The use of particles constrained to multiple rings allows the computational time required for one Monte Carlo sweep with a large number of vortices to be reduced from $O(N^2)$ to $O(N)$.

It is clear that the time required to complete one Monte Carlo sweep of N free particles must grow as $O(N^2)$ – each sweep is N attempted moves, and the attempt at each move requires a computation of an energy change which (if one is unable to make use of fast multipole-type methods) requires the calculation of $N - 1$ interaction terms. Consequently the cost grows prohibitively expensive to find extremal states for very large numbers of particles.

It is shown by Bergensen et al [2] and by Lim [10] et al [11] the problem of logarithmic interaction by point vortices on the surface of the sphere takes on the mean field approximation [10] as the number of points grows infinitely large. The distribution of vortices becomes one nearly uniform in area over the sphere.

Experimentally such a simplification appears justified: it is seen that for the system as computed with N free particles, or for $\frac{N}{2}$ rings of two particles each, or $\frac{N}{3}$ rings of three particles each, or up to $\sqrt{N}$ rings of $\sqrt{N}$ vortices each, the equilibria found all have approximately equal energies.

If the number of vortices N is a perfect square one may then form $\sqrt{N}$ rings each of $\sqrt{N}$ vortices uniformly spaced along lines of latitude. This results in only $\sqrt{N}$ particles being moved and thus being tracked in the Monte Carlo algorithm – so one full sweep requires only $\sqrt{N}$ attempted moves, and each move requires on the order of $\sqrt{N}$ pairwise calculations to determine system energy. In total, a single sweep of this reduced system requires only $O(N)$ computations. Experimentally the state found by this reduction of order tends to be close to the extremal configuration, with an energy agreeing very closely to the lowest energy found by treating all particles as independent points. Thus the order of the system has been reduced to $O(N)$.

2.5 Ring Configurations

Several non-ground state equilibria can be found if one recalls the note that some relative equilibria for the problem of vortices on the surface of the sphere are latitudinal rings – a collection of m evenly spaced vortices of uniform strength arranged on one line of latitude. Lim, Montaldi, and Roberts [14] classify many one- and two-ring relative equilibria. By making the assumption that interesting states will be found which are built of several latitudinal rings a slight modification of the Monte Carlo algorithm as used in the above section allows the detection of several new energy minimums, not generally the global energy minimum.

The Monte Carlo algorithm is modified in these ways: rather than track n vortices, one tracks a single representative vortex from each of n rings, and assume that m vortices are evenly spaced along the line of latitude of each of these tracked vortices. The representative vortices are allowed to move over the surface of the sphere just as they would be in the 'completely free' case in which no latitudinal rings existed, and the other vortices in that ring are moved by the same amount in the same direction (relative the north-south axis). As with the regular routine the change in energy and in the angular momentum is computed and the same Metropolis rule algorithms are applied to the system.

Otherwise the method is equivalent to the above algorithm. The initial placements are done at random, and the Monte Carlo iterator run with an extremely high value of the inverse temperature β for a very long run. The representative vortices align themselves into a configuration which minimizes the energy subject to the constraint that the system consists of n rings of m vortices each.

For few particles, such as 4, 6, and 8 particle systems, the two-ring state is the global energy minimum or 'ground' state. For systems with a prime number of vortices, the placing of all vortices in one ring results in a single equatorial ring, as one might predict – this is the configuration that maximizes the distance each vortex is from its neighbors subject to the constraint that all are on the same line of latitude. For numbers of particles which may be wholly divided into an odd number of rings several interesting phenomena occur, notably that the system places at least one ring onto the equator, and then balances an equal number of latitudinal rings in the northern and the southern hemisphere.

2.6 Radial Distribution Function

A term borrowed from the statistical mechanics of gases, the radial distribution function represents the density of particles, relative to a uniform density, at a given radius from any given reference point [6]. In this paper the use of such a function is straightforward – the vortex particles behave in some ways like a gas (hence the 'vortex gas' label for such a model)– and so it serves as a tool by which the structure of the system can be measured.

The radial distribution function is defined for a given angular separation θ. (On the surface of the unit sphere, this is equivalent to the separation by a distance θ.) In the studies of gas and crystal propeties from which the notion of a radial distribution function is derived, the radial distribution function is taken to be the number of pairs separated by a given range of distances, divided by the number of pairs which would be expected in a completely uniform distribution – typically meaning one divides the number of particles in a shell by a uniform density times the volume of that shell [6]. For this paper that normalization is not used.

Definition 1 *For a set of particles $\vec{x}_j, j = 1...N$ define the radial distribution function to be the set of ordered pairs*

$$D = \{(\theta, n) |\ n \text{ is the number of pairs of points with separation } \theta\} \quad (4)$$

– that is, θ is a radial separation with n pairs of points with that separation.

In a disorganized system the radial distribution function tends to 1 after an initial peak [6]. In a crystalline system with considerable structure, individual isolated peaks appear with the radial distribution function tending towards zero in the regions between those peaks [6]. Therefore one expects direct observation of the radial distribution function to suffice for determining when the system has reached a regular patterned equilibrium pattern. In practice one does observe several large peaks in the radial distribution function when the equilibrium is reached.

In this paper graphs of the radial distribution functions are simplified by grouping together angles which are $\Delta\theta = \frac{\pi}{180}$ apart, making the plots represent the numbers of vertices per degree apart.

2.7 Recognition of Patterns

The recognition and classification of patterns on the surface of the sphere is in principle straightforward.

Definition 2 *Two patterns of points on the sphere are equivalent if a combination of rotations and reflections in $O(3)$ make the positions of one pattern coincident to the positions of the other.*

The definition is intuitively clear and suggests the problem of verifying whether two patterns are identical can be done visually, which for problems with few vortices can be done. Nevertheless checking all possible rotations and reflections of a pattern becomes excessively time-consuming as soon as more than a few vortices are under consideration.

A simpler fingerprint to compute and to use in comparisons remains a practical necessity. A satisfactory tool for this task is the radial distribution function as described in section 2.6.

Theorem 3 *The radial distribution function $D(\vec{z})$ is a property of the group orbit under $O(3)$ actions.*

Proof. The radial distribution function is a measure of the angles between points in a given configuration. $O(3)$ actions rotate and reflect these points on the surface of the sphere but do not change the relative separations of any of these points. Therefore the radial distribution functions will be identical for all elements in the orbit of a given configuration $\vec{z}$. □

Corollary 4 *Two vortex configurations cannot be equivalent unless their radial distribution functions agree.*

Proof. If a configuration $\vec{z}_2$ is equivalent to state $\vec{z}_1$ there exists a combination of rotations and reflections – an action in $O(3)$ – which turns $\vec{z}_1$ into $\vec{z}_2$. Therefore the radial distribution function for configuration $\vec{z}_2$ must be equal to that of configuration $\vec{z}_1$. □

There is a caveat necessary to observe. While two equivalent configurations must have identical radial distribution functions, it is not proven that two configurations with identical radial distribution functions must be equivalent.

The radial distribution function also presents an interesting means of studying any pairwise interaction such as a Coulomb potential. Consider an energy-like potential function $U(\vec{z})$ for a collection of points z with uniform strength.

Theorem 5 *The value of $U(\vec{z})$, for a system in which all particles have uniform strength, is uniquely determined by the radial distribution function of $\vec{z}$.*

Proof. The value of the energy-like potential function $U(\vec{z})$ is defined by

$$U(\vec{z}) = \sum_{i=1}^{n} \sum_{j=i}^{n} s_i s_j f(\vec{z}_i, \vec{z}_j) \tag{5}$$

with $f(\vec{z}_i, \vec{z}_j)$ a pairwise distance-dependent function. Assuming without loss of generality that each particle has strength $s_i = 1$, then the value of this function is:

$$\begin{aligned} U(\vec{z}) &= \sum_{i=1}^{n} \sum_{j=i}^{n} f(\vec{z}_i, \vec{z}_j) && (6) \\ &= \sum_{i=1}^{n} \sum_{j=i}^{n} g(d(\vec{z}_i, \vec{z}_j)) && (7) \\ &= \sum_{d_i \in D} n(d_i) g(d_i) && (8) \end{aligned}$$

where the functions f and g are as above, d is the scalar distance function, and $n(d)$ is the number of pairs separated by the distance d.

From the construction of the energy-like potential function as the summation over the elements in the radial distribution function from equation 8 it is obvious this property of the points $\vec{z}$ uniquely defines the value of $U(\vec{z})$, regardless of whether any two configurations points $\vec{z}_1$ and $\vec{z}_2$ with the same radial distribution function are congruent placements of points. □

3 Free and Ring Equilibria On The Sphere

3.1 Known Polyhedron Shapes

Although the general problem of optimally placing points on the surface of the sphere is unsolved, there are known to exist several regular polyhedrons with vertex positions which may be exactly calculated. The correspondence between the configurations of these regular shapes and the Monte Carlo computations of vortex locations provides confirmation that this Monte

Carlo method will produce vortex configurations consistent with analytically proven results.

The polyhedron vertex configurations as developed by Coxeter [4] have a preferred direction – showing symmetry, at least, around the z-axis, and often symmetry around the other axes. The Monte Carlo method has no such preferred direction. However, comparison of the radial distribution function for polyhedron configurations with those of the Monte Carlo-derived configurations indicates at least several of the numerically derived shapes are the same configuration as those polyhedron vertices.

3.2 Ring Notation

It is convenient to have a shorthand method of describing the order in which rings appear on the surface of the sphere, beginning with a north pole which is either occupied by a lone vortex, or which is along an axis of rotational symmetry for the vortex configuration, and continuing southward along lines of latitude until all the vortices in a given configuration are accounted for. For this we use a description like this:

$$(i, j^s, k^a) \tag{9}$$

In this shorthand i, j, and k represent latitudinal rings of i, j, or k (respectively) evenly spaced vortices, with the first ring the most northern, and the final ring the most southern. There may be any number of latitudinal rings, including 'rings' of a single vortex of necessity (by the requirements of rotational symmetry) at the north or south poles.

The superscript in j^s means the vortices on the ring of j vortices are staggered with respect to those of the latitudinal ring north of it: the line of longitude on which a vortex in the second ring occurs bisects the lines of longitudes of the two nearest vortices on the first ring. The superscript in k^a means the vortices on the ring of k vortices are aligned with respect to those of the latitudinal ring north of it: the lines of latitude for the vortices in the third ring are identical to those of the vortices in the second.

Dividing the relation between two rings into 'staggered' or 'aligned' seems of obvious use only when two rings have the same number of vortices. The use can be extended to systems in which one ring has a whole multiple of the number of vortices in its predecessor. By convention for this case, a pair of rings described $(i, 2i^a)$ means half the vortices in the second ring are on lines of longitude for vortices of the first ring. The notation $(i, 2i^s)$ means none of the lines of longitude for vortices in the second ring are lines of longitude for vortices in the first.

Though this vocabulary may seem extremely limited it is sufficient to describe all the equilibrium configurations discovered over the course of this paper, and as well to describe configurations representing the five Platonic solids and a few of the Archimedean solids.

4 Numerical Results

Examined in this section is the agreement between the results of the Monte Carlo algorithm and the configuration of several known polyhedron vertices. Initial successes in matching the configurations of points for shapes such as the tetrahedron and the octahedron suggest regular polyhedrons may represent ground states for the problem of vortices on the surface of the sphere. This is however not the case, as examination of the problem with 8 vortices demonstrates – the rectangular antiprism is a shape with a lower energy than the cube has. Nevertheless several interesting configurations can be developed by considering the case of N free particles and then the case of several rings of fewer particles (for nonprime N). So below are outlined analytical results for some known regular shapes and the vortex problem with different constraints on the number of latitudinal rings.

The numerical results derived here, for the problems with up to 20 particles or rings evaluated, were run for 1,000 sweeps of the particles or rings in the case, with an inverse temperature β of 100,000,000. The maximum angle by which any vortex was allowed to move was 0.05 radians. The random number seed used was the time of day, and the calculations performed by computers running Matlab (and reliant upon its built-in random number sequence generator). The source codes are available upon request or at the web site http://www.rpi.edu/∼limc/laboratory.html.

The first subsection of many of these configurations is the result not of a Monte Carlo simulation, but rather the result of using the precise coordinates of the vertices of a polyhedron as derived by Coxeter [4]. The use of these shapes for reference validates the routines which calculate system energy, radial distribution function, and which plot the positions of these points. The maps of the plots are an unprojected latitude-longitude system, unfortunately distorting to an extent the angles between vortex pairs. In practice this leaves the maps misleading or confusing only for the problems in which free particles are considered – those which consider latitudinal rings are forced to be rotationally symmetric around the z axis, which makes the unprojected latitude-longitude map easier to accurately understand.

Each subsection after the first – apart from the section considering the case of $N = 12$, for which two polyhedrons with analytically located vertices are considered – is the result of a Monte Carlo simulation.

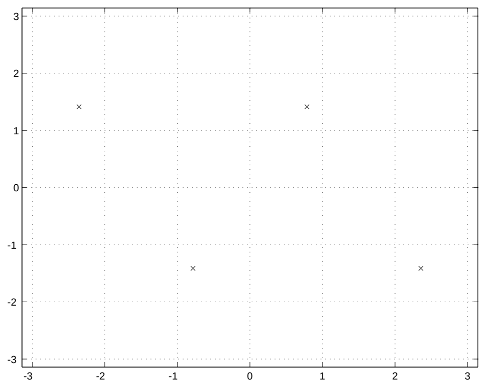
4.1 Four Vortices

4.1.1 Tetrahedron

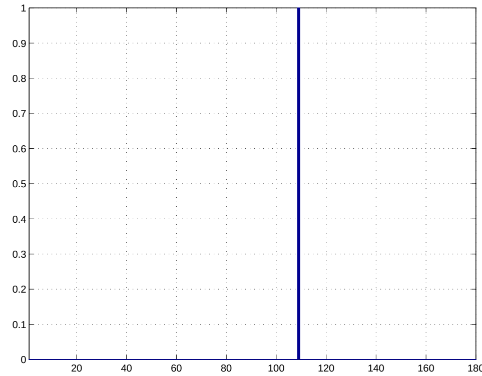
The tetrahedron is one of the Platonic solids [4] and is a commonly recognized shape. In the vortex problem it is a dynamic equilibrium.

The tetrahedron can be considered as a configuration of either two rings of two vortices each, staggered with respect to one another (so that the line of longitude passing through one ring's vortices bisects the angle between two adjacent vortices of the opposite ring) and arranged symmetrically about the origin. Using the shorthand of section 3.2 it may be written $(2, 2^s)$, meaning one ring of two vortices and then another rings of two vortices staggered with respect to the one preceding it. It may also be regarded as a single ring of three evenly spaced vortices with one polar vortex. The shorthand for this configuration would be $(1, 3)$, a single point necessarily at a pole and a ring of three evenly spaced vortices on one line of longitude. That there are multiple ring notations for the same configuration is not rare.

Exact State Energy: -1.72609243471069



Latitude-Longitude Map
Figure 4.1.1a



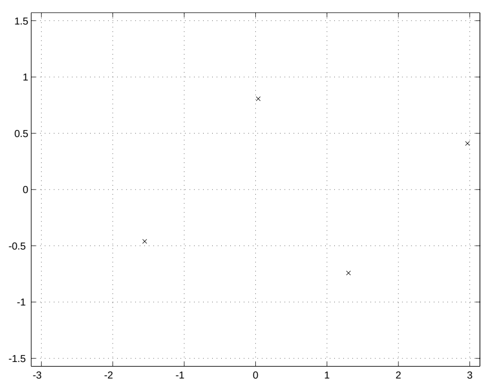
Radial Distribution Function
Figure 4.1.1b

4.1.2 Free Particles

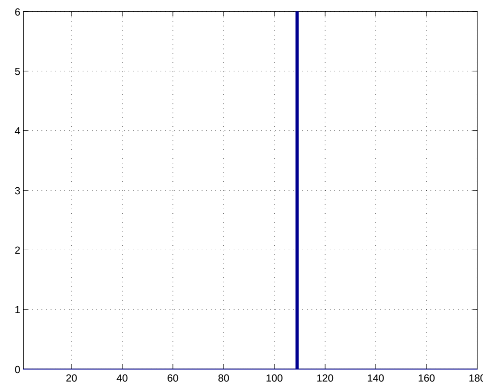
Four completely free particles find the tetrahedron configuration. As the Monte Carlo algorithm has no means of detecting latitude in the case of free particles the configuration will be a tetrahedron 'pointed' in a randomly chosen direction. Because of this, and the distortion of angles and areas forced by projecting the surface of the sphere onto the latitude-longitude rectangle plotted, the tetrahedron is not obvious in that plot (as several other shapes will not be obvious from their plots, as in sections 4.3.2 and 4.5.2). The energy and the radial distribution function indicate the tetrahedron has been found, and if one examines a plot of the vortex points on the surface of the sphere a rotation which brings it to match Coxeter's angles (section 4.1.1) can be found, and examining a plot on the surface of the sphere reinforces that conclusion.

This configuration of vortices, as it is, does not lend itself to being described with the shorthand notation of section 3.2. The configuration can be rotated until either $(2, 2^s)$ or $(1, 3)$ describe it.

Monte Carlo State Energy: -1.72609240448660



Latitude-Longitude Map
Figure 4.1.2a

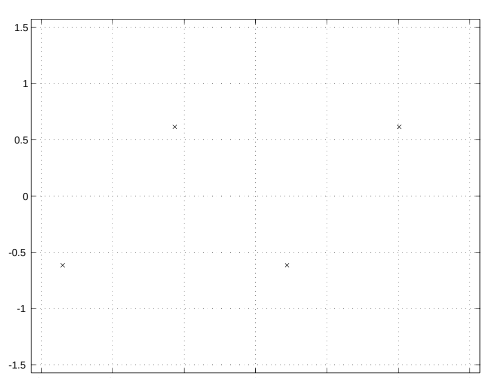


Radial Distribution Function
Figure 4.1.2b

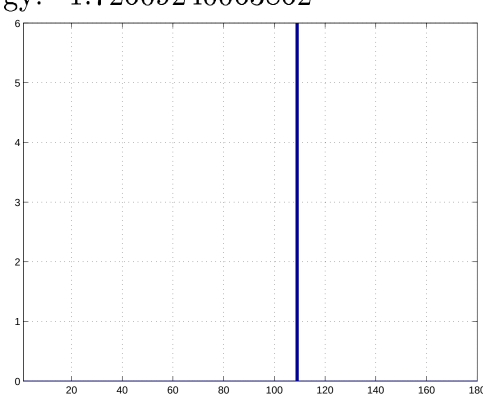
4.1.3 Two Rings

Two rings of two vortices each produce a tetrahedron again. This shape looks like Coxeter's (section 4.1.1) rather than the free particle case (section 4.1.2) as the constraint to latitudinal rings forces the Monte Carlo algorithm to pick a configuration which is symmetric around the z axis. The configuration has one ring as far north of the equator as the other is south, with the longitudinal positions of the rings staggered, a pattern which will reappear many times as equilibra are found. It is also obviously the $(2, 2^s)$ configuration.

Monte Carlo State Energy: -1.72609240063862



Latitude-Longitude Map
Figure 4.1.3a

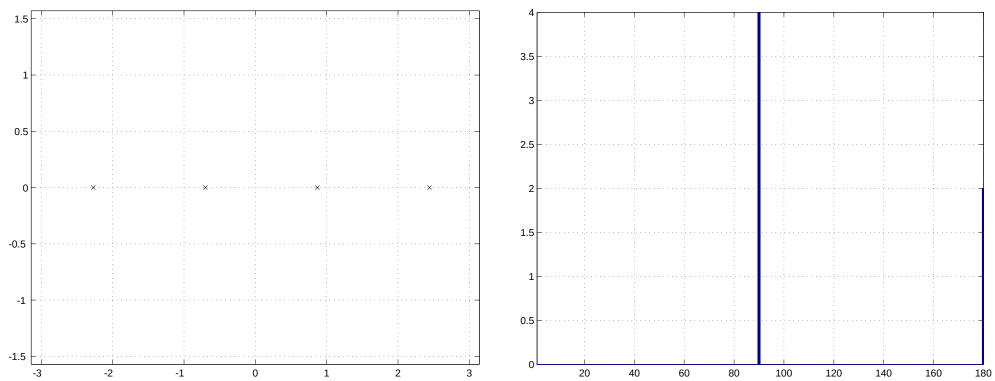


Radial Distribution Function
Figure 4.1.3b

4.1.4 One Ring

With all the vortices constrained to a single latitudinal ring, the only equilibrium state is to equally space all four around the equator. In practice any great circle, if covered with evenly spaced vortices, is an equilibrium, but the constraint to points on a latitudinal ring leave only one great circle available. This is a state unlikely to be found if the system is not constrained because its energy is greater than that of the tetrahedron. This configuration can be described as (4).

Monte Carlo State Energy: -1.38629436094795



Latitude-Longitude Map
Figure 4.1.4a

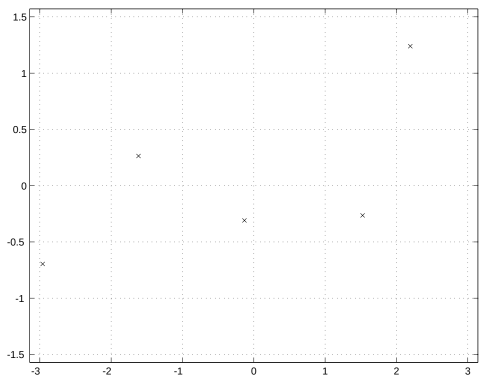
Radial Distribution Function
Figure 4.1.4b

4.2 Five Vortices

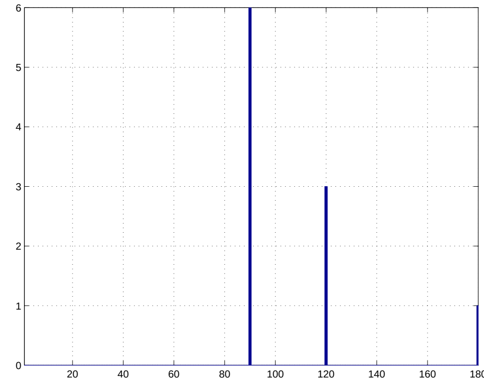
4.2.1 Free Particles

Examined for the sake of completeness is the case of five vortices. This is the first configuration encountered which cannot be expressed as a collection of rings with a uniform number of vortices in each ring. It can still be described in the shorthand ring notation, however, as the configuration $(1, 3, 1)$ – two polar rings and a latitudinal ring of three vortices, on the equator, if the single vortices are considered the north and south poles. The ring of three vortices is not (generally) in the plane with $z = 0$, the result of there being no preferred direction for free particles. (For latitudinal rings there is a preferred direction – all configurations must be rotationally symmetric around the z-axis – but the free particles problem does not have this constraint.) This configuration is the vertices of the double tetrahedron.

Monte Carlo State Energy: -1.90954242989071



Latitude-Longitude Map
Figure 4.2.1a

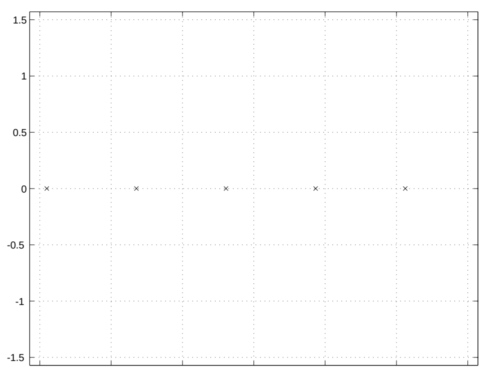


Radial Distribution Function
Figure 4.2.1b

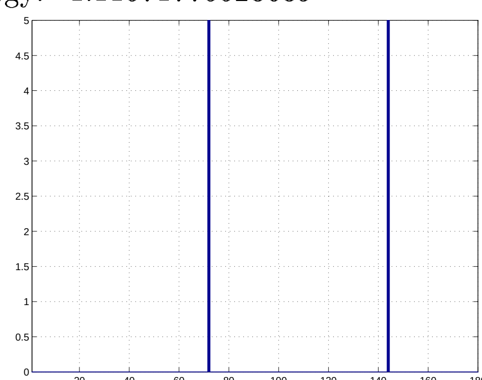
4.2.2 One Ring

A single ring of five vortices – the configuration (5) – is an equilibrium as can be easily verified algebraically, but as the energy of this configuration is greater than that of the free particles it is not a ground state for the system of five particles. It is the ground state for five particles when all are constrained to the same line of latitude, which gives rise to an observation regarding its stability as a dynamic equilibrium: as a rigid body with all these vortices constrained to a single line of latitude the system is stable. As a system of five free vortices this is not stable [17].

Monte Carlo State Energy: -1.11571775528689



Latitude-Longitude Map
Figure 4.2.2a



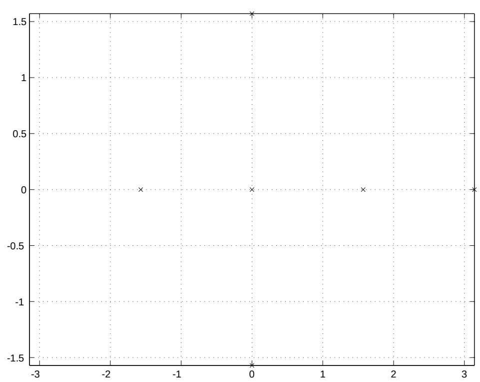
Radial Distribution Function
Figure 4.2.2b

4.3 Six Vortices

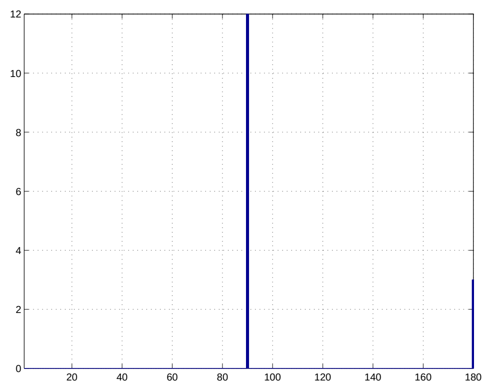
4.3.1 Octahedron

The octahedron, another Platonic solid, lends itself to two ring notations. By fixing vortices to the north and south pole the notation $(1, 4, 1)$ describes this shape. It can also be viewed as the configuration $(3, 3^s)$, a triangular of vortices in the northern hemisphere and another, staggered, triangular arrangement of vortices in the southern hemisphere.

Exact State Energy: -2.07944154167984



Latitude-Longitude Map
Figure 4.3.1a

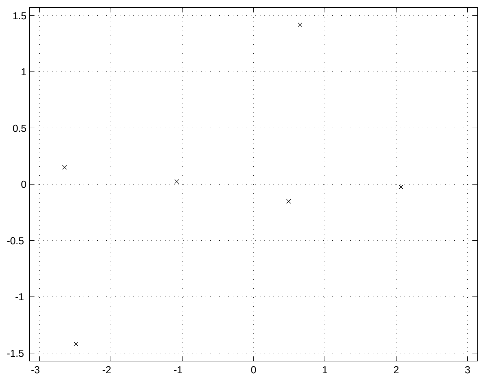


Radial Distribution Function
Figure 4.3.1b

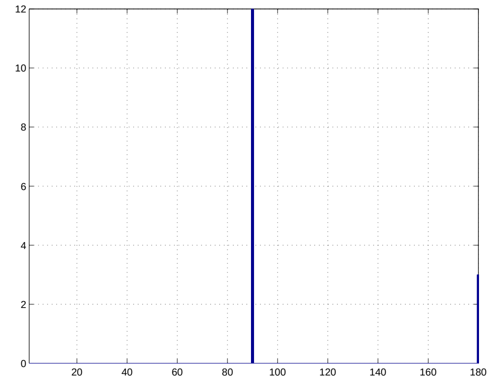
4.3.2 Free Particles

The case of six free particles tends towards the ground state for any six vortices on the sphere, which is the octahedron configuration seen in section 4.3.1. As with four free particles (section 4.1.2) the projection onto the plane distorts the appearance, obscuring the $(1, 4, 1)$ or the $(3, 3^s)$ shape, but the radial distribution function points it out. The system energy is also equal to that of the octahedron configuration, indicating the octahedron is most likely the ground state for six vortices.

Monte Carlo State Energy: -2.07944143162953



Latitude-Longitude Map
Figure 4.3.2a

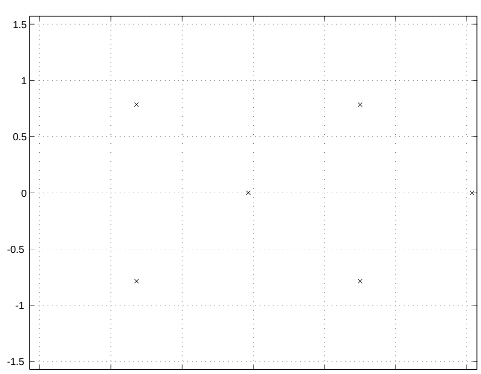


Radial Distribution Function
Figure 4.3.2b

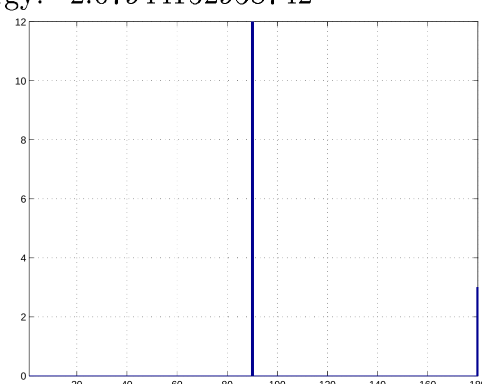
4.3.3 Three Rings

By constraining the problem to three rings of two vortices each one might expect a new configuration. This proves not the case. Consideration of the energy of the $(2, 2^s, 2^s)$ arrangement of vortices – equal to the ground state (sections 4.3.1 and 4.3.2) – and the identical radial distribution function to the established octahedron pattern indicate one should expect this is the octahedron again. Drawing the solution on the surface of the sphere confirms this pattern has been rediscovered.

Monte Carlo State Energy: -2.07944152958742



Latitude-Longitude Map
Figure 4.3.3a

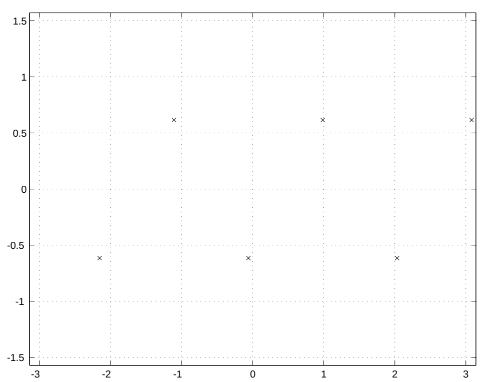


Radial Distribution Function
Figure 4.3.3b

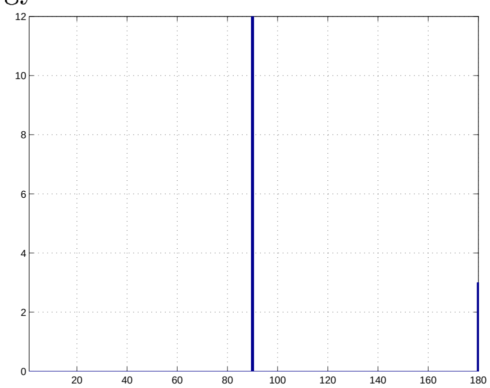
4.3.4 Two Rings

It has already been argued (section 4.3.1) that the octahedron configuration may be written as $(3, 3^s)$. From this it is not surprising when the equilibrium found is $(3, 3^s)$ and the system energy and radial distribution function match that of the octahedron.

Monte Carlo State Energy: -2.07944153808856



Latitude-Longitude Map
Figure 4.3.4a

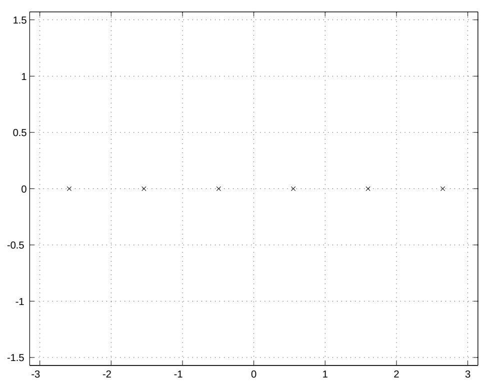


Radial Distribution Function
Figure 4.3.4b

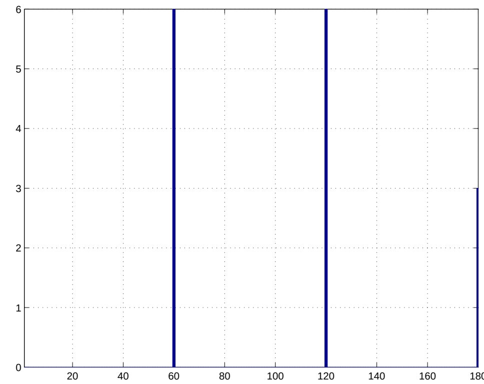
4.3.5 One Ring

A single equatorial ring of six vortices has a higher system energy than the octahedron, so it is not the ground state for the six vortex problem. It is still an equilibrium – an equatorial ring of evenly spaced vortices is [14] – and is the ground state for the problem in which all the vortices are constrained to be on the same line of latitude. In consequence while six vortices on the equator is not stable for free particles [17], it is for the constrained problem.

Monte Carlo State Energy: -0.35334910492587



Latitude-Longitude Map
Figure 4.3.5a



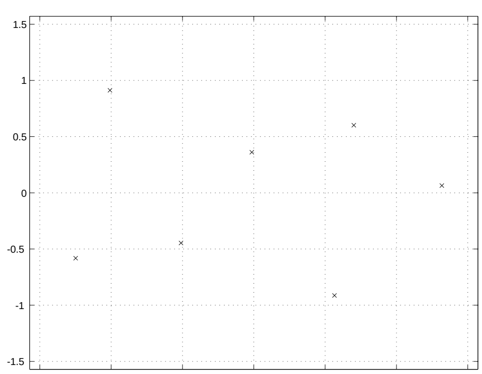
Radial Distribution Function
Figure 4.3.5b

4.4 Seven Vortices

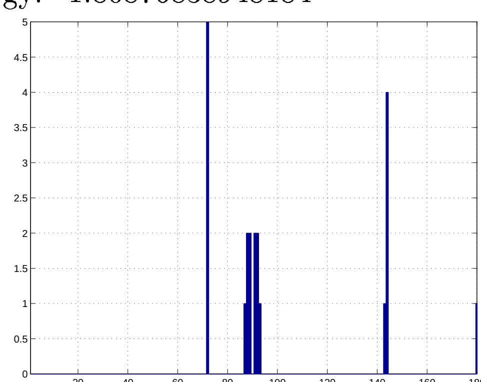
4.4.1 Free Particles

As with the case of five free particles (section 4.2.1) this equilibrium cannot be written as a set of several rings with the same number of vortices in each ring. Also like the case of five free particles the configuration can be written as a northern and southern vortex (not parallel to the z-axis in this drawing as the Monte Carlo routine has nothing to orient it in the free particle case, but which can be made parallel the z-axis by the proper O(3) rotation) and an equatorial ring. The configuration is $(1, 5, 1)$.

Monte Carlo State Energy: -1.80876838948184



Latitude-Longitude Map
Figure 4.4.1a

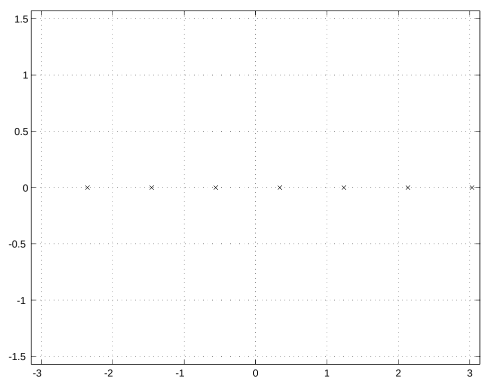


Radial Distribution Function
Figure 4.4.1b

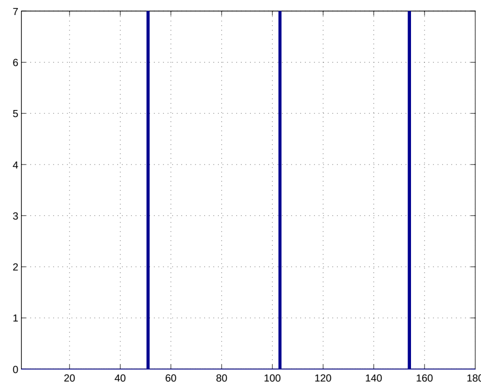
4.4.2 One Ring

As in sections 4.1.4, 4.2.2, and 4.3.5 one can place all seven vortices uniformly spaced along the equator to reach an equilibrium. This configuration has higher energy than the other known equilibrium, and is known to not be a stable equilibrium if all the vortices are allowed to move freely [17], but it is stable if the vortices are constrained to all lie on the same line of latitude.

Monte Carlo State Energy: 0.93471975410475



Latitude-Longitude Map
Figure 4.4.2a



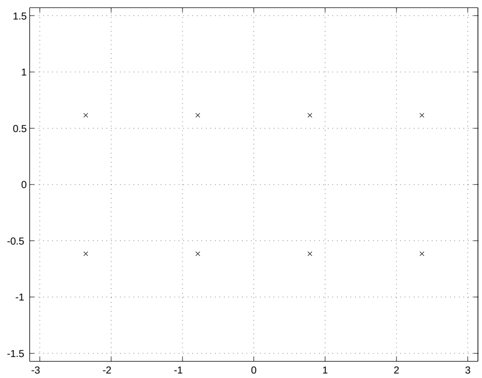
Radial Distribution Function
Figure 4.4.2b

4.5 Eight Vortices

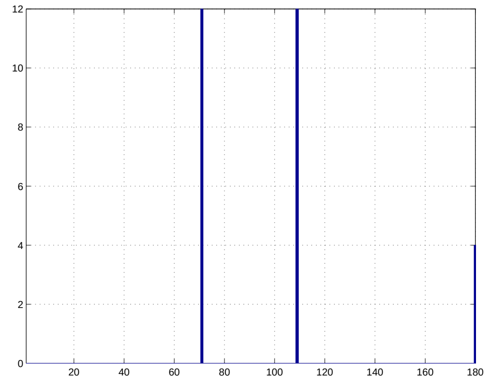
4.5.1 Cube

The cube is another Platonic solid. Unlike the tetrahedron and octahedron (sections 4.1.1 and 4.3.1) this configuration does not lend itself to being written as a pair of staggered rings. The most immediately obvious configuration is $(4, 4^a)$. It can be written, if one places points on the poles, by the configuration $(1, 3, 3^s, 1)$. It is interesting then that this first Platonic solid which cannot be written solely as a collection of staggered rings without placing points on the poles (or which can be written with aligned rings) is not the ground state for the case of eight vortices. The rectangular antiprism, $(4, 4^s)$ is.

Exact State Energy: -1.35919229436318



Latitude-Longitude Map
Figure 4.5.1a

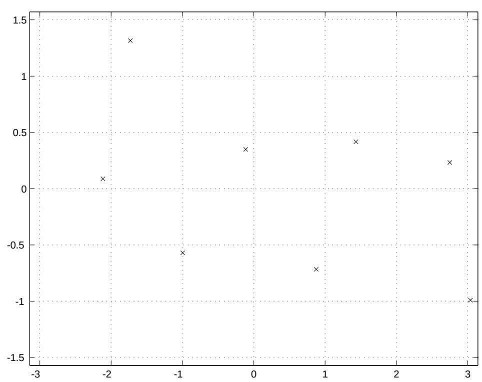


Radial Distribution Function
Figure 4.5.1b

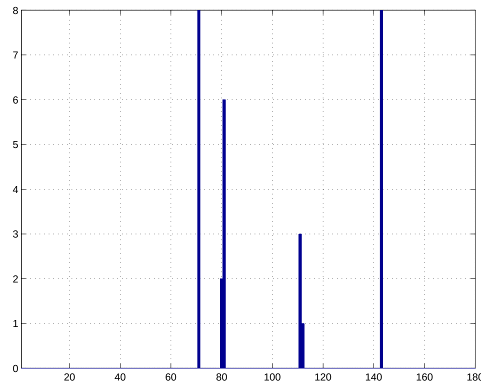
4.5.2 Free Particles

Eight free particles finds the ground state for this system. The shape found – not at all clear on the flat, projected, map – is that of the rectangular antiprism, with its configuration described by $(4, 4^s)$. The energy is noticeably lower than that of the cube. This is the same configuration detected by four rings of two vortices each (section 4.5.3) and of two rings of four vortices each (section 4.5.4), although the radial distribution functions look very slightly different. This is the result of numerical variations – vortices very nearly 80 degrees apart (for example) falling slightly below or slightly above the exact angle. This results in the heights and widths of the peaks in the radial distribution functions varying slightly (compare figures 4.5.2b, 4.5.3b, and 4.5.4b), but examination of these functions indicate they represent substantially equal configurations.

Monte Carlo State Energy: -1.44791448814153



Latitude-Longitude Map
Figure 4.5.2a

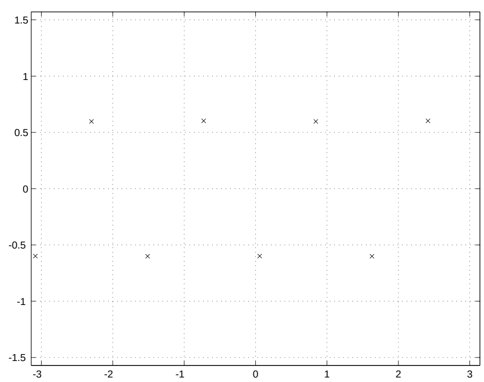


Radial Distribution Function
Figure 4.5.2b

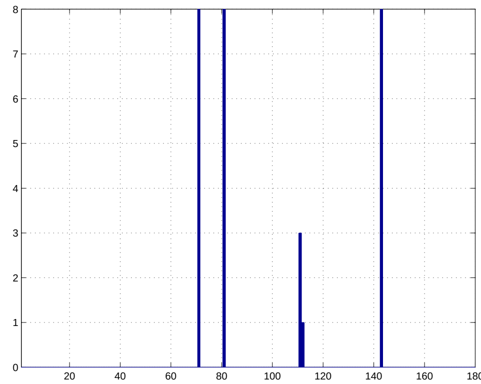
4.5.3 Four Rings

Four rings of two particles each collapse neatly into a single northern ring and a single southern ring, for the $(4, 4^s)$ configuration characteristic of the rectangular antiprism. It is, from the energy, the ground state, and its radial distribution function is recognizably close to that in figures 4.5.2b and 4.5.4b. The difference is minute fluctuations in the angles between the several rings. While the Monte Carlo algorithm can get extremely close to a 'perfect' configuration the tiny moves of rings needed to fine-tune the positions remain unlikely, as only a very small move in a narrow range of directions will improve the configuration.

Monte Carlo State Energy: -1.44791447942788



Latitude-Longitude Map
Figure 4.5.3a

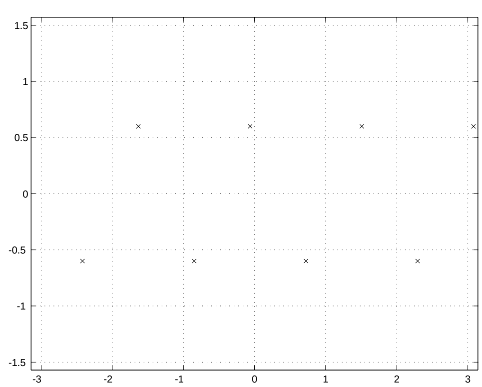


Radial Distribution Function
Figure 4.5.3b

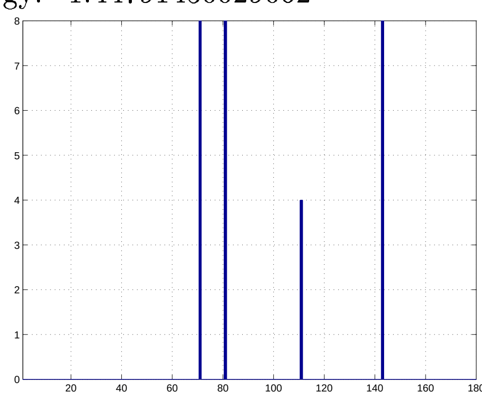
4.5.4 Two Rings

Two rings of four vortices each – again the ground state – offers the best match between a Monte Carlo derived calculation and the exact rectangular antiprism. The radial distribution function does not appear to have any of the numerical fluctuation seen in figures 4.5.2b or 4.5.3b, while the energy remains near the ground state. This is the best $(4, 4^s)$ configuration derived by these probabilistic methods.

Monte Carlo State Energy: -1.44791450029062



Latitude-Longitude Map
Figure 4.5.4a

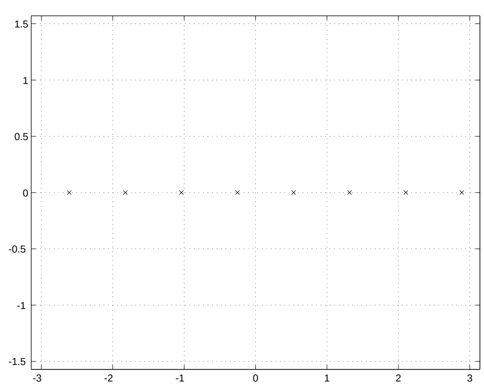


Radial Distribution Function
Figure 4.5.4b

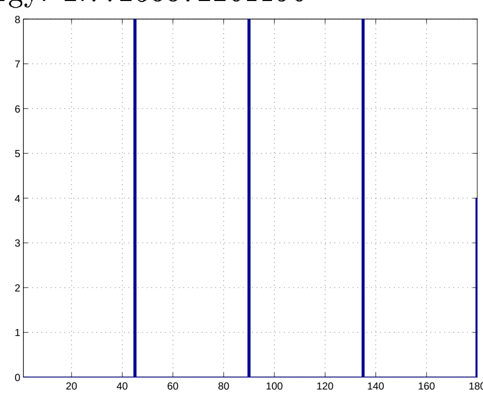
4.5.5 One Ring

This single ring of eight vortices – configuration (8) – has an energy higher than the ground state, just as the single equatorial ring of six vortices (section 4.3.5) and of four vortices (section 4.1.4) did. A clear pattern (obvious after consideration) emerges in the radial distribution function (figure 4.5.5b), with N vortices in this single ring seeing $\frac{N}{2}$ peaks evenly spaced between 0 and π radians of separation. The first $\frac{N}{2} - 1$ of them have N vortices each and the final peak (at an angular separation of π radians) has $\frac{N}{2}$ pairs with that separation. This pattern holds for all the problems of a single equatorial ring.

Monte Carlo State Energy: 2.77258872261196



Latitude-Longitude Map
Figure 4.5.5a



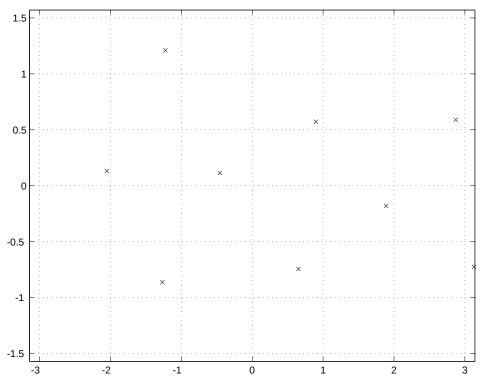
Radial Distribution Function
Figure 4.5.5b

4.6 Nine Vortices

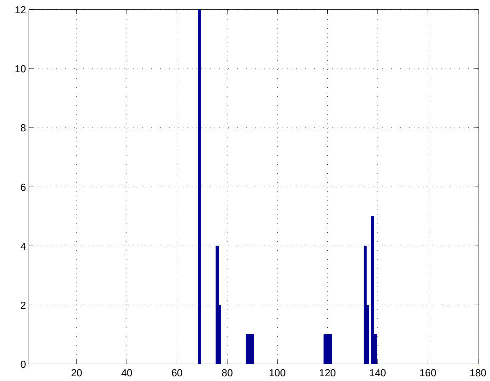
4.6.1 Free Particles

The configuration of nine free particles does not appear, from a latitude-longitude mapping, to be any particular configuration. This is one of several instances in which plotting the positions on the surface of a sphere and being able to rotate the sphere is indispensible to understanding the output. By rotating the sphere on which these points are plotted the configuration $(3, 3^s, 3^s)$ appears – this is several staggered rings of three vortices each.

Monte Carlo State Energy: -0.82214934597115



Latitude-Longitude Map
Figure 4.6.1a



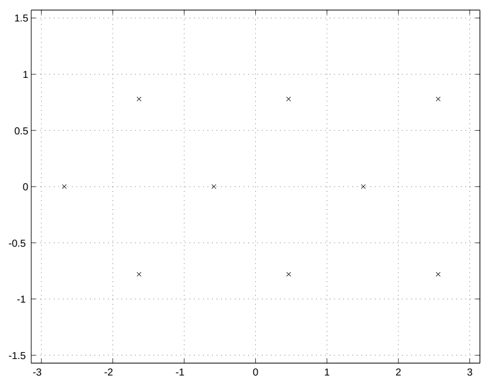
Radial Distribution Function
Figure 4.6.1b

4.6.2 Three Rings

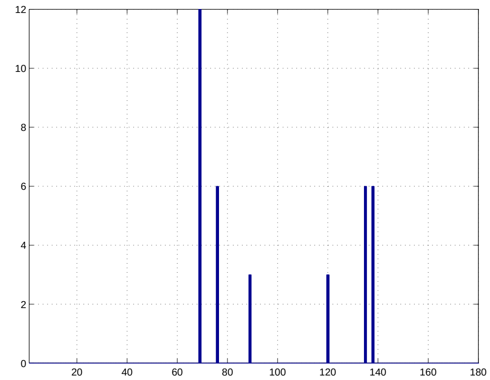
The case of three rings with three vortices each settles into the configuration $(3, 3^s, 3^s)$, and its energy and radial distribution function indicate that it is the free particle configuration as seen in section 4.6.1. The radial distribution function is sharper, with higher narrower peaks, than it is for figure 4.6.1b, the result of there being fewer different independent pairs of particles interacting in the system – an effect which will be seen again in this paper.

Of possible significance is that nine vortices is the first case considered to date in which an odd composite number of particles has been considered. As with the cases of even particles, the ground state is reached either with all the particles left free or with the particles grouped into rings each with the smallest nontrivial divisor of nine vortices.

Monte Carlo State Energy: -0.82220677484952



Latitude-Longitude Map
Figure 4.6.2a

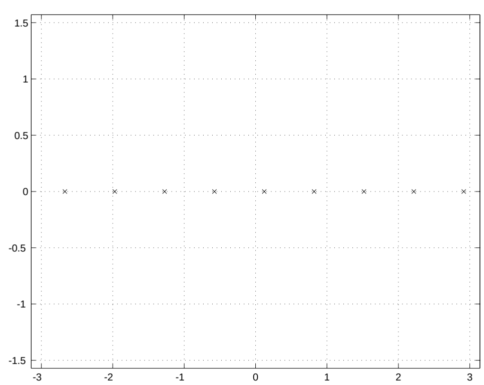


Radial Distribution Function
Figure 4.6.2b

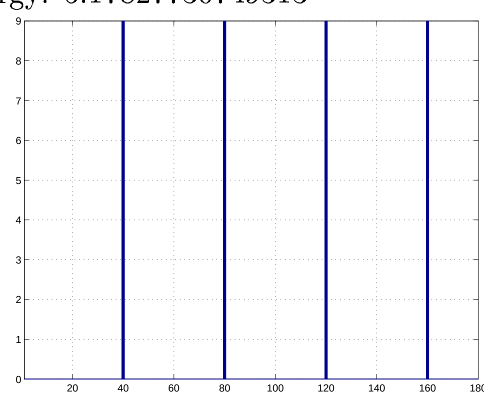
4.6.3 One Ring

As has been noted (in sections 4.1.4, 4.2.2, et cetera) the vortices placed in an evenly spaced configuration along the equator is an equilibrium, with the highest energy found for an equilibrium of nine vortices.

Monte Carlo State Energy: 5.17827730749313



Latitude-Longitude Map
Figure 4.6.3a



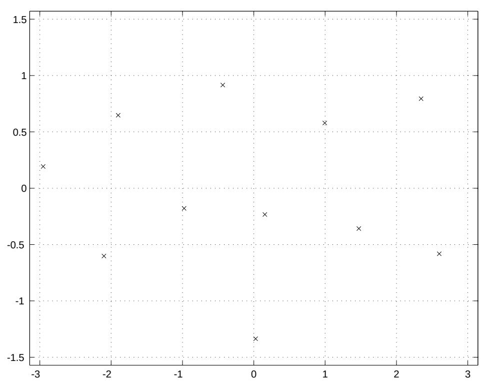
Radial Distribution Function
Figure 4.6.3b

4.7 Eleven Vortices

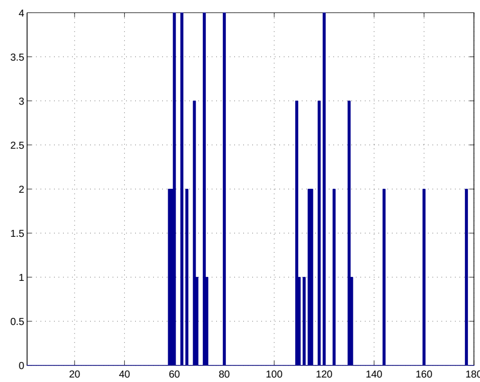
4.7.1 Free Particles

As with the cases of five and seven free particles (sections 4.2.1 and 4.4.1) this equilibrium cannot be written as a set of several rings with the same number of vortices in each ring. However, it can be written as a case with one polar vortex (assumed without loss of generality to be the north pole) and two rings with equal numbers of vortices: the configuration $(1, 5, 5^s)$. As with the diagram in section 4.6.1, the latitude-longitude map does not present clearly the structure of this equilibrium, but it can be found by a rotation of the surface of the sphere.

Monte Carlo State Energy: 1.28406980046918



Latitude-Longitude Map
Figure 4.7.1a

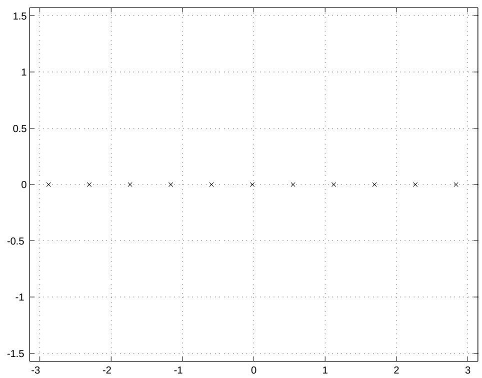


Radial Distribution Function
Figure 4.7.1b

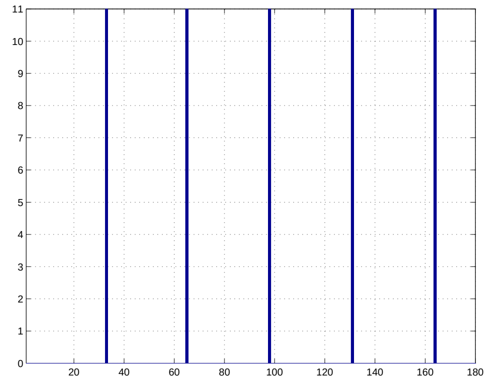
4.7.2 One Ring

As in sections 4.1.4, 4.2.2, and 4.3.5 one can place all eleven vortices uniformly spaced along the equator to reach an equilibrium. The energy of this state is much higher than that in section 4.7.1 and the configuration is not a stable equilibrium for free particles [17], though for the particles constrained to a latitudinal ring it is.

Monte Carlo State Energy: 11.74624693398740



Latitude-Longitude Map
Figure 4.7.2a



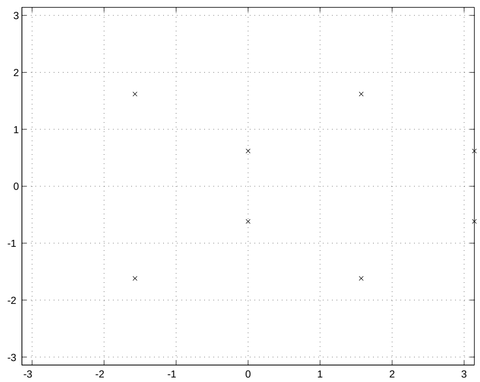
Radial Distribution Function
Figure 4.7.2b

4.8 Twelve Vortices

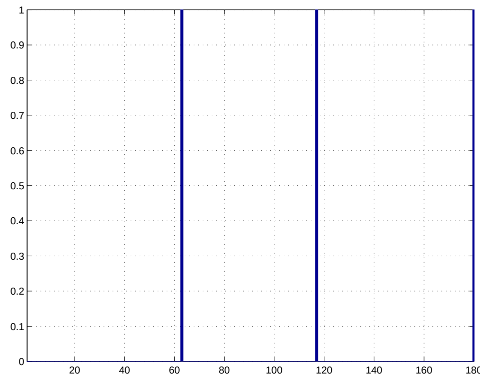
4.8.1 Icosahedron

With twelve vortices one has two polyhedron configurations as provided by Coxeter's array of Platonic solids and their duals [4]. The first of these is the icosahedron and it can be written in the configuration $(1, 5, 5^s, 1)$. It may also be written in the configuration $(3, 3^s, 3^s, 3^s)$ (as seen in section 4.8.5). This staggered configuration, it is demonstrated in section 4.8.3, is the ground state for the system of twelve vortices. The icosahedron is not the only configuration of twelve vortices with analytically known points; another configuration, the cuboctahedron, is described in section 4.8.2 and can be derived from latitudinal rings.

Exact State Energy: 2.53542345606662



Latitude-Longitude Map
Figure 4.8.1a

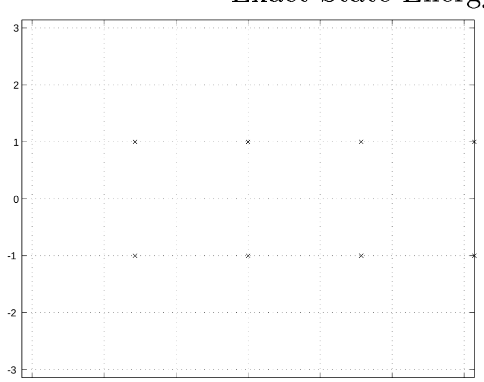


Radial Distribution Function
Figure 4.8.1b

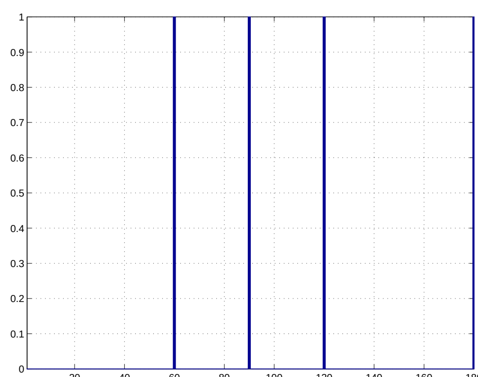
4.8.2 Cuboctahedron

The cuboctahedron is not a Platonic solid, but is one of the Archimedean solids [18]. It can be represented as three rings of four each, with the configuration $(4, 4^s, 4^s)$. This shape is not the ground state for the 12-vortex problem. Were it not for the example of the vertices of the cube not being the ground state for the eight-vortex problem, one might be tempted to suppose the vertices of a Platonic solid represent ground states for that many vortices on the surface of the spheres.

Exact State Energy: 2.74548665548307



Latitude-Longitude Map
Figure 4.8.2a

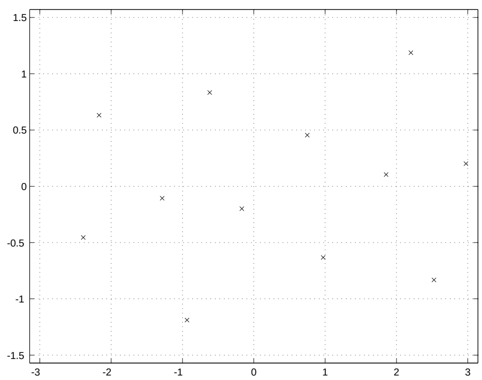


Radial Distribution Function
Figure 4.8.2b

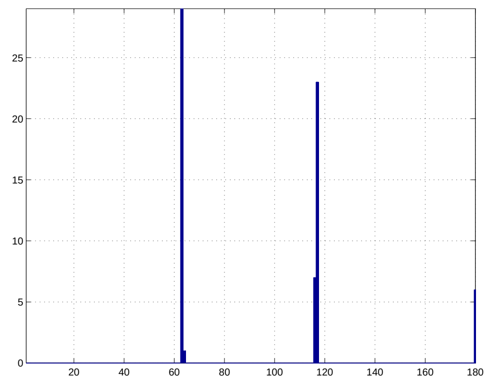
4.8.3 Free Particles

The ground state for the twelve particle case is found, by the energy and radial distribution function, to be the icosahedron (section 4.8.1). As with the case of eight free particles (section 4.5.2) and the case of four rings of two vortices each (section 4.5.3) the radial distribution function is slightly displaced from the ideal configuration, reflecting numerical variation around the icosahedron configuration. This state is $(3, 3^s, 3^s, 3^s)$.

Monte Carlo State Energy: 2.53542350723281



Latitude-Longitude Map
Figure 4.8.3a

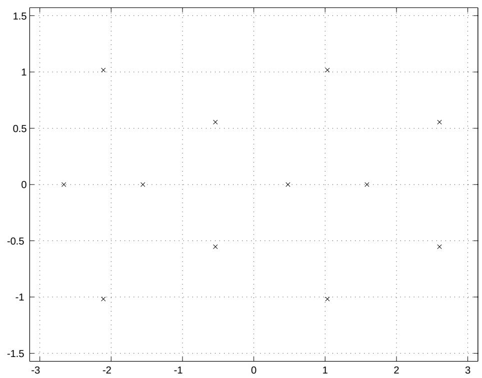


Radial Distribution Function
Figure 4.8.3b

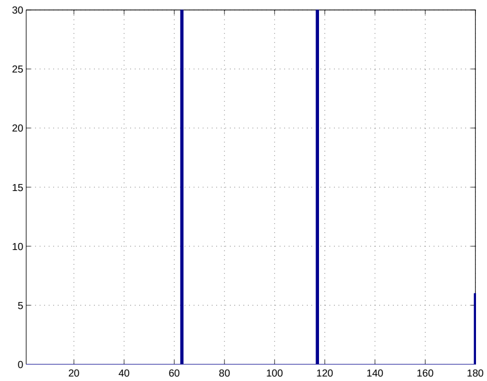
4.8.4 Six Rings

This configuration is the ground state. From its energy and radial distribution function it is a rotation of the icosahedron (section 4.8.1). Its configuration by ring structures is $(2, 2^s, 4^s, 2^s, 2^s)$.

Monte Carlo State Energy: 2.53542354364253



Latitude-Longitude Map
Figure 4.8.4a

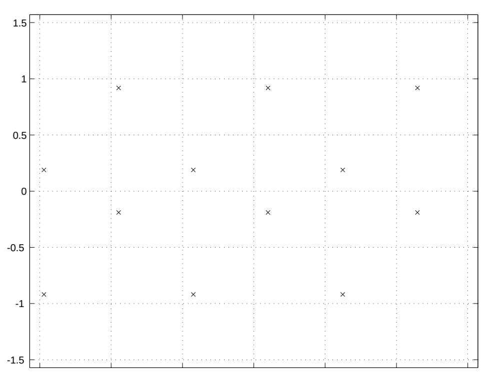


Radial Distribution Function
Figure 4.8.4b

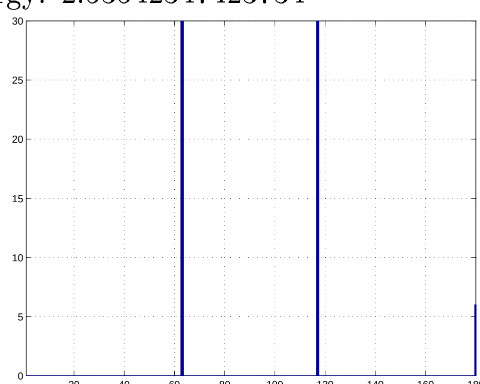
4.8.5 Four Rings

Four rings of three vortices each reaches the ground state again. its configuration is $(3, 3^s, 3^s, 3^s)$, duplicating the already found configurations for the icosahedron.

Monte Carlo State Energy: 2.53542347423734



Latitude-Longitude Map
Figure 4.8.5a

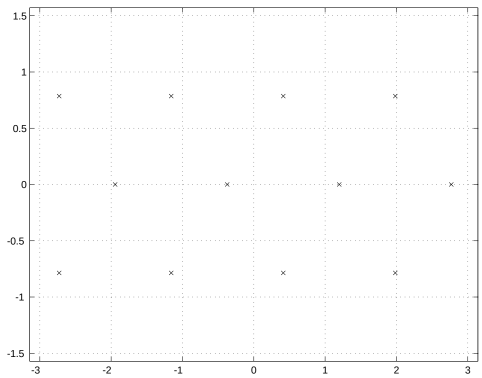


Radial Distribution Function
Figure 4.8.5b

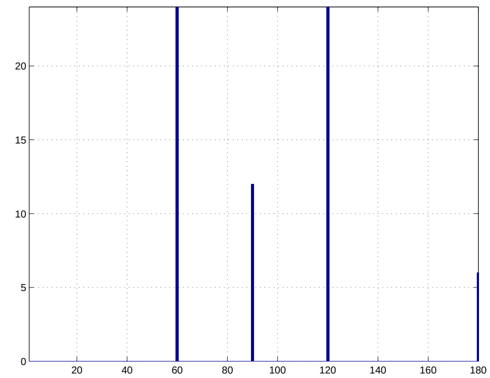
4.8.6 Three Rings

Three rings of four vortices each, the configuration $(4, 4^s, 4^s)$, again does not reach the ground state. It is the cuboctahedron instead.

Monte Carlo State Energy: 2.74548667907740



Latitude-Longitude Map
Figure 4.8.6a

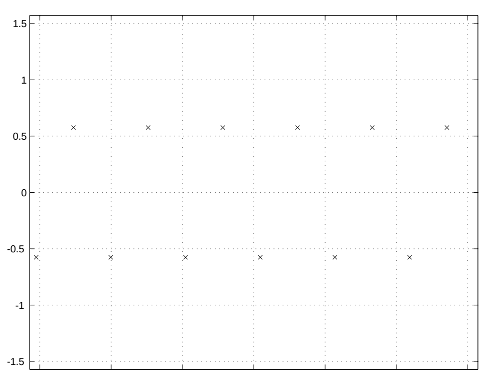


Radial Distribution Function
Figure 4.8.6b

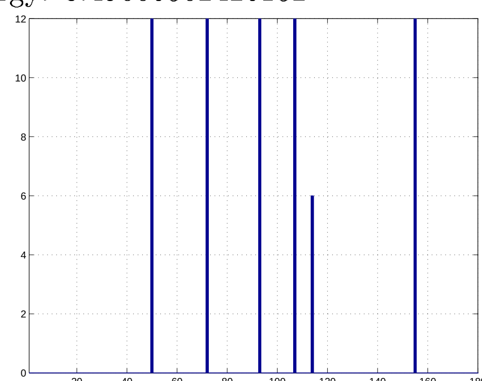
4.8.7 Two Rings

Two rings of six vortices each is a state which is neither the ground state nor the cuboctahedron. The $(6, 6^s)$ configuration does not have another ring shape derived by rotating it. For the first time (in this paper) a state has been found in which the two-ring solution is not the ground state. It is, however, a familiar equilibrium [14], the staggered equilibrium. It is not stable as a system of free particles [14] but as two latitudinal rings of six vortices each it is.

Monte Carlo State Energy: 3.49006502426152



Latitude-Longitude Map
Figure 4.8.7a

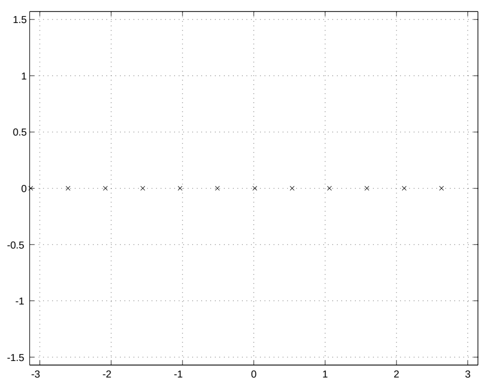


Radial Distribution Function
Figure 4.8.7b

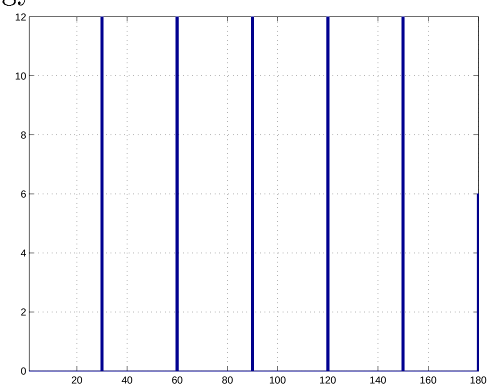
4.8.8 One Ring

The (12) configuration of an equatorial ring with twelve vortices is the highest energy state found by the Monte Carlo methods of this paper. That trait of very high energy is common to the equatorial rings: the energy of the system is proportional to the proximity of vortices, and the uniformly spaced equatorial ring packs the vortices more closely than the multiple-ring configurations do.

Monte Carlo State Energy: 15.92883412057909



Latitude-Longitude Map
Figure 4.8.8a



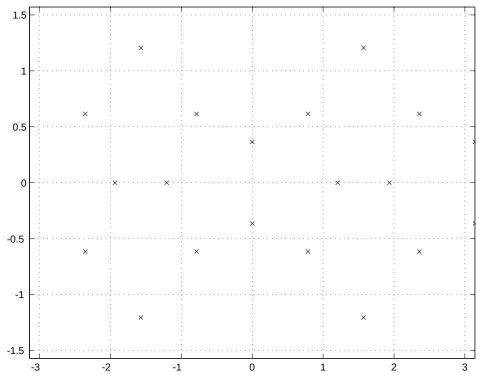
Radial Distribution Function
Figure 4.8.8b

4.9 Twenty Vortices

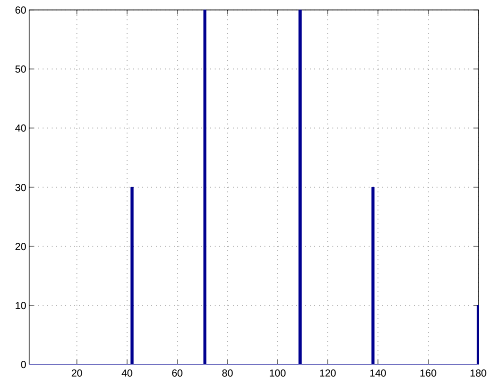
4.9.1 Dodecahedron

The dodecahedron becomes possible once one has twenty vortices. It is again one of the Platonic solids, with a ring configuration $(5, 5^a, 5^s, 5^a)$. This is the second Platonic solid (after the cube, section 4.5.1) to not be the ground state, from the results of the following section 4.9.2.

Exact State Energy: 24.46341682027342



Latitude-Longitude Map
Figure 4.9.1a

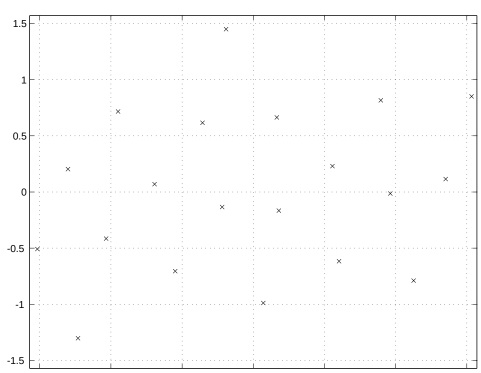


Radial Distribution Function
Figure 4.9.1b

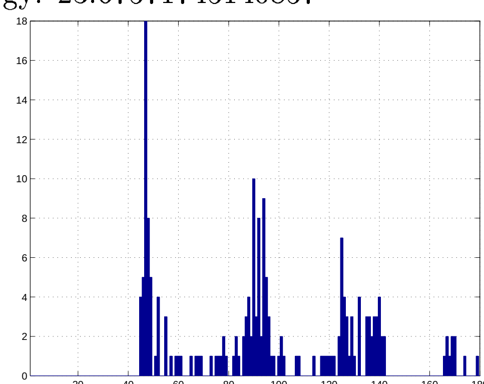
4.9.2 Free Particles

Twenty free particles find the ground state, the configuration $(5, 5^s, 5^s, 5^s)$. This is not the dodecahedron configuration. The radial distribution function becomes here somewhat shaky, the result of more particles in the system. Longer run times reduce the 'blurring' of this radial distribution function, but relatively slowly.

Monte Carlo State Energy: 23.67571745140857



Latitude-Longitude Map
Figure 4.9.2a

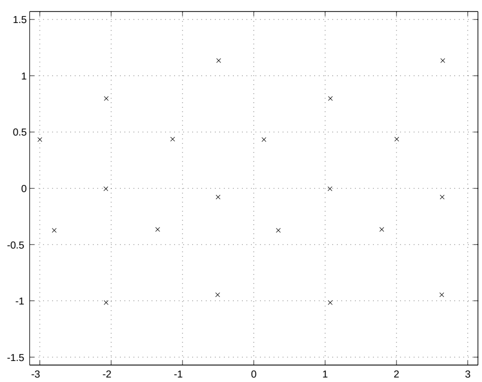


Radial Distribution Function
Figure 4.9.2b

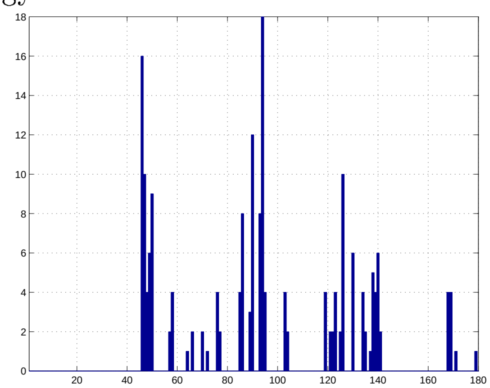
4.9.3 Ten Rings

The configuration of ten rings of two vortices each – $(2, 2^s, 4^s, 2^s, 2^s, 4^s, 2^s, 2^s)$ – does not, on an initial glance, obviously resemble that of the free particles of section 4.9.2. However, the peaks in the radial distribution function for this case are in the same positions and of approximately the same heights as those in figure 4.9.2b. This and the extremely close match between the system energies indicate these are the same configuration.

Monte Carlo State Energy: 23.67765462136305



Latitude-Longitude Map
Figure 4.9.3a

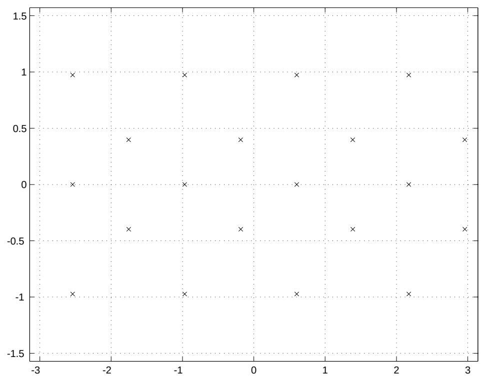


Radial Distribution Function
Figure 4.9.3b

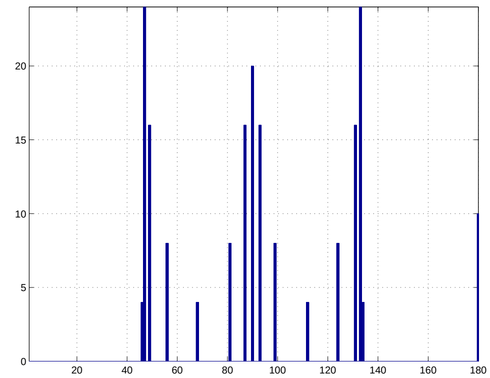
4.9.4 Five Rings

While five rings with four vortices each – the configuration $(4, 4^s, 4^s, 4^s, 4^s)$ – has an energy only slightly greather than those in the ten rings of two vortices (section 4.9.3) or free vortices (section 4.9.2) – the radial distribution functions are strikingly different, moreso than would be expected from variations around the dodecahedron shape. From examining the points as they appear on the surface of the sphere the configurations are not identical. This is the first case in this paper in which complex configurations can be shown to be different not by examining energy but by considering the radial distribution function, validating its use as an identification tool.

Monte Carlo State Energy: 23.70789835283543



Latitude-Longitude Map
Figure 4.9.4a

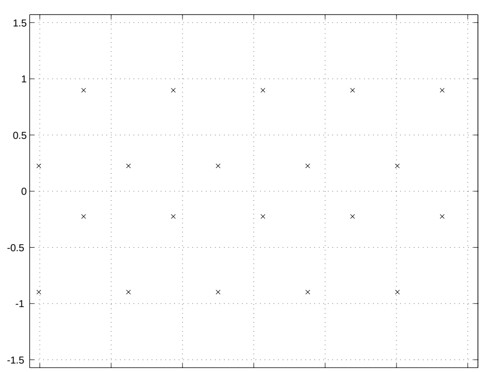


Radial Distribution Function
Figure 4.9.4b

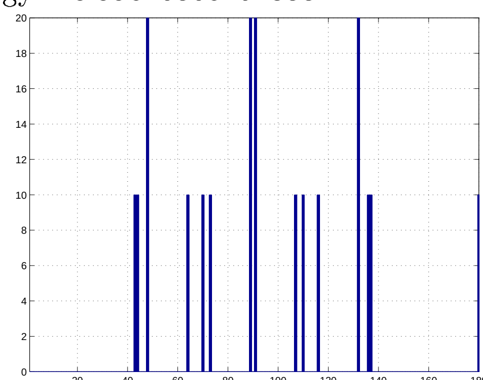
4.9.5 Four Rings

Four rings of five vortices each, $(5, 5^s, 5^s, 5^s)$, is once more a new state, with an energy noticeably higher than the case of five rings with four vortices each, and with a radial distribution function very different to match.

Monte Carlo State Energy: 23.99016860254888



Latitude-Longitude Map
Figure 4.9.5a

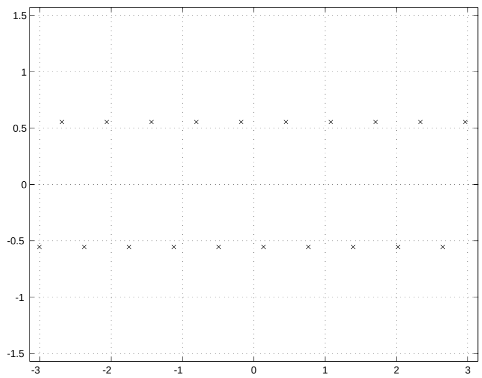


Radial Distribution Function
Figure 4.9.5b

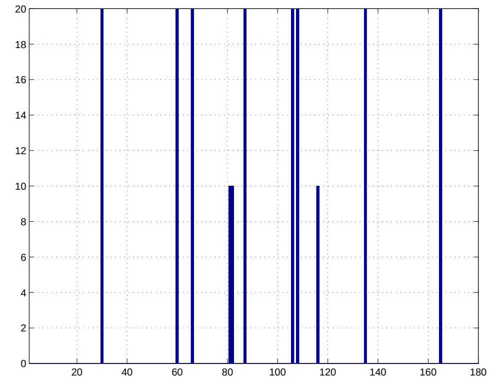
4.9.6 Two Rings

Two rings of ten vortices each, $(10, 10^s)$, is is another known equilibrium state [14], with a higher energy than the other states again. The radial distribution function is nearly symmetric around $\frac{\pi}{2}$ radians, reflecting the very symmetric arrangement of points.

Monte Carlo State Energy: 30.26690609911227



Latitude-Longitude Map
Figure 4.9.6a

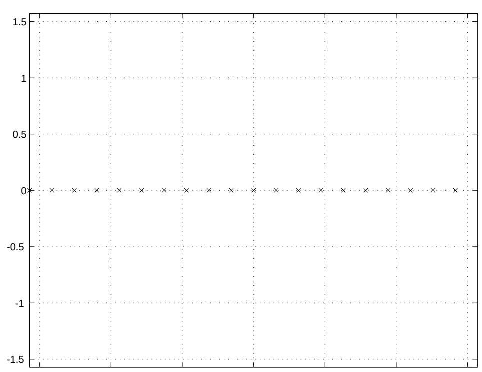


Radial Distribution Function
Figure 4.9.6b

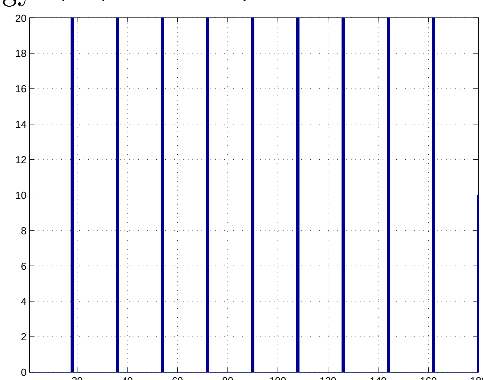
4.9.7 One Ring

Once again the highest energy state for N particles is found by arranging them all on the equator. The (20) configuration also shows the uniform radial distribution function noted in section 4.8.8.

Monte Carlo State Energy: 71.783318841718\u200b84



Latitude-Longitude Map
Figure 4.9.7a



Radial Distribution Function
Figure 4.9.7b

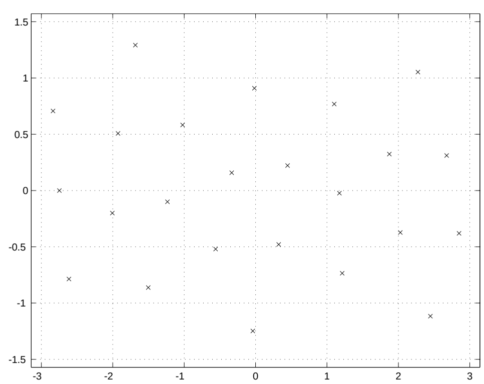
4.10 Twenty-Four Vortices

4.10.1 Free Particles

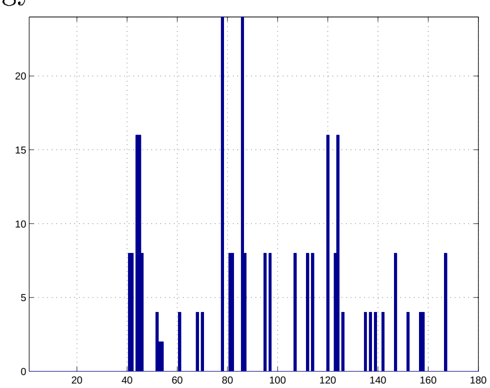
Twenty-four vortices are not represented by a Platonic solid or an immediately related shape as developed by Coxeter [4]. It is examined to validate the system for numbers of vortices not understood by the study of Platonic or related solids. The ground state has a configuration of $(4, 4^s, 4^s, 4^s, 4^s, 4^s)$ – matched by the system of twelve rings of two vortices each (section 4.10.2) and six rings of four vortices each (section 4.10.4).

For this problem, with a great number of particles in motion, the standard run time of 1,000 sweeps was extended to 10,000 sweeps was used instead, with the intent of making the approximation to the ground state better.

Monte Carlo State Energy: 40.88652829900479



Latitude-Longitude Map
Figure 4.10.1a

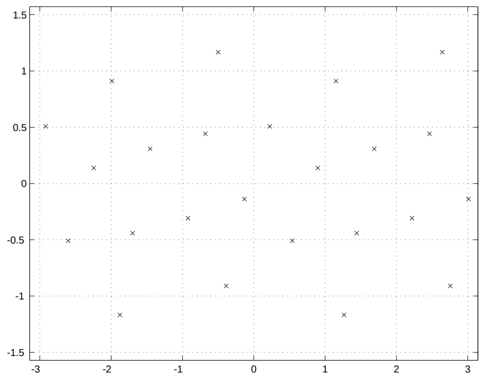


Radial Distribution Function
Figure 4.10.1b

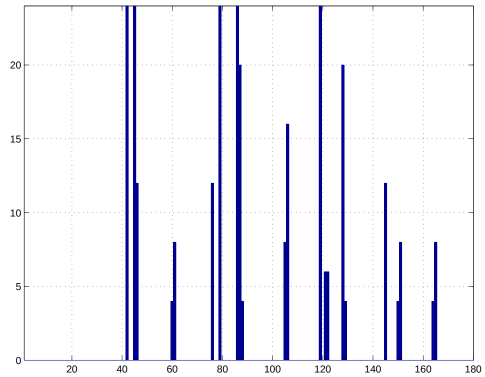
4.10.2 Twelve Rings

¿From the system energy and the radial distribution function one concludes this case is equivalent to the 24 free particles (section 4.10.1) and six rings of four vortices each (section 4.10.4). As in section 4.10.1 the results used are from a run of 10,000 sweeps rather than just 1,000.

Monte Carlo State Energy: 40.88072018342196



Latitude-Longitude Map
Figure 4.10.2a

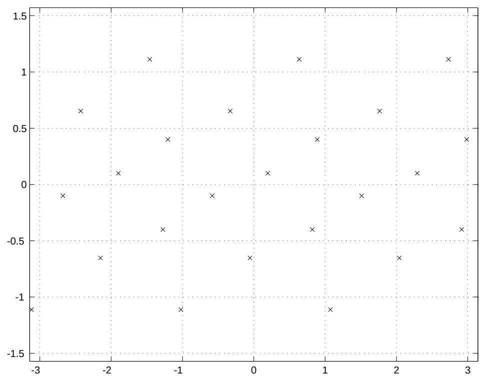


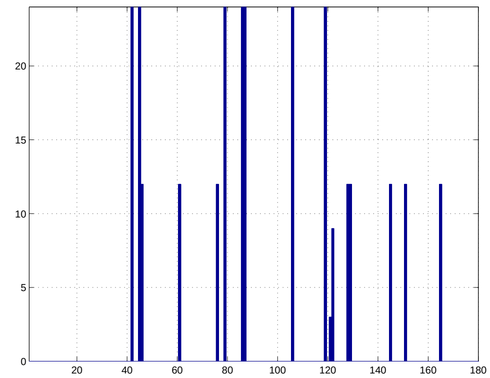
Radial Distribution Function
Figure 4.10.2b

4.10.3 Eight Rings

The energy is the first striking data point for the case of eight rings of three vortices each. This $(3, 3^s, 3^s, 3^s, 3^s, 3^s, 3^s, 3^s)$ configuration almost exactly matches the free particles (section 4.10.1), twelve rings of two vortices (section 4.10.2), and six rings of four vortices (section 4.10.4). The radial distribution function for this shape looks initially quite different, the result of the vortices nearly $\frac{\pi}{2}$ apart fluctuating more than about a degree out of their best positions. This case has about 48 vortex pairs between 88 and 92 degrees apart; the case of six rings of four vortices each has just as many in the same interval. The apparent difference is principally an artifact of the axes on which the radial distribution functions are drawn. As in sections 4.10.1 and 4.10.2 the run was 10,000 sweeps.

Monte Carlo State Energy: 40.88065360028251





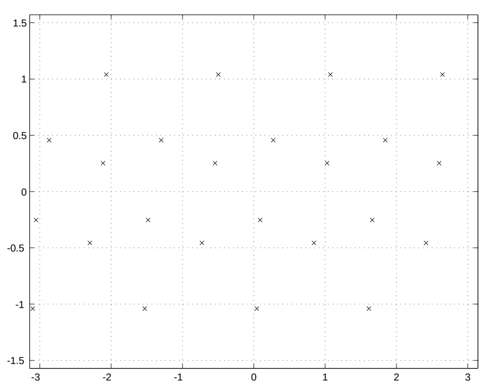
Latitude-Longitude Map
Figure 4.10.3a

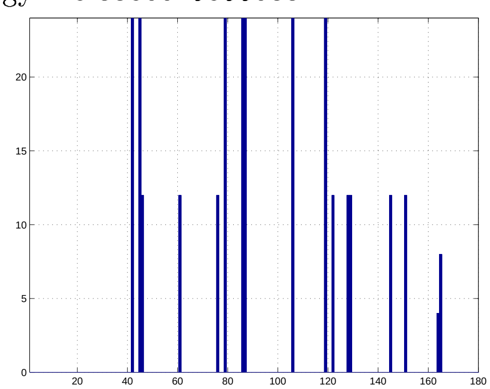
Radial Distribution Function
Figure 4.10.3b

4.10.4 Six Rings

As outlined in the preceding sections (4.10.1, 4.10.2, 4.10.3) this configuration is the ground state again, represented this time as $(4, 4^s, 4^s, 4^s, 4^s, 4^s)$. The difficulties of getting many particles into precise positions – which would eliminate the fuzziness in the radial distribution function as plotted – is highlighted here, as alternating rows would be expected to settle onto the same line of longitude. The accumulated variations around the ultimate ground state leave them slightly out of that ideal. As above a run of 10,000 sweeps was used for these results.

Monte Carlo State Energy: 40.88066450556884





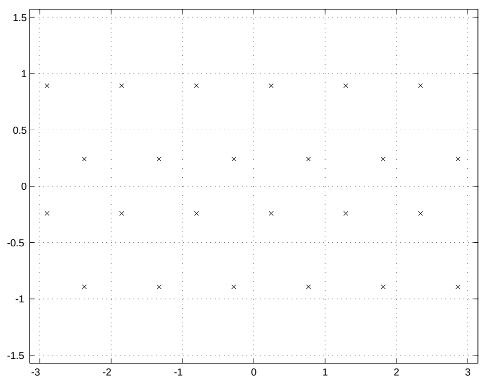
Latitude-Longitude Map
Figure 4.10.4a

Radial Distribution Function
Figure 4.10.4b

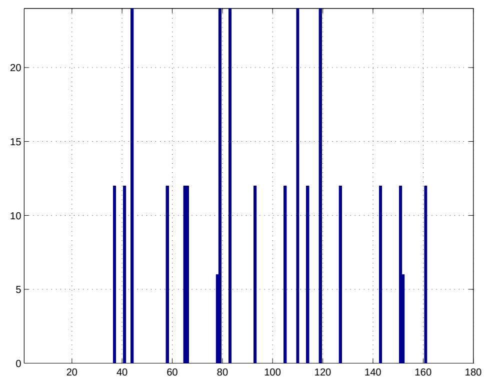
4.10.5 Four Rings

The four rings of six vortices each case, $(6, 6^s, 6^s, 6^s)$, finally detects a new configuration. The energy is higher and the radial distribution function is markedly different in the placement of its peaks. These results are derived from only the typical run of 1,000 sweeps.

Monte Carlo State Energy: 41.64018355305313



Latitude-Longitude Map
Figure 4.10.5a

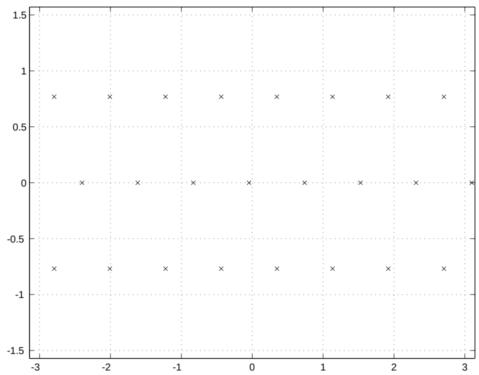


Radial Distribution Function
Figure 4.10.5b

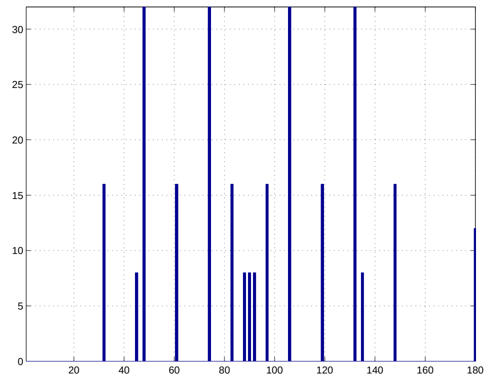
4.10.6 Three Rings

The energy grows higher yet for the $(8, 8^s, 8^s)$ case. This state happens to demonstrate a common property to problems with an odd number of rings: one ring tends toward the equator, and an even number of rings settle into the northern and southern hemispheres.

Monte Carlo State Energy: 43.22833069732373



Latitude-Longitude Map
Figure 4.10.6a

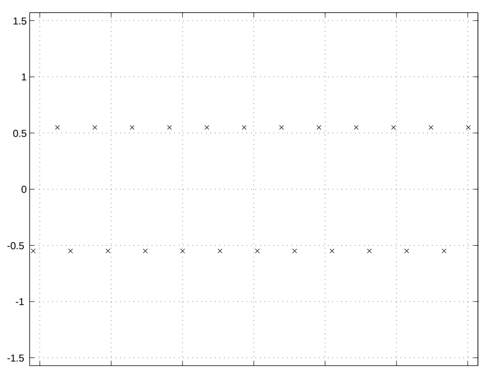


Radial Distribution Function
Figure 4.10.6b

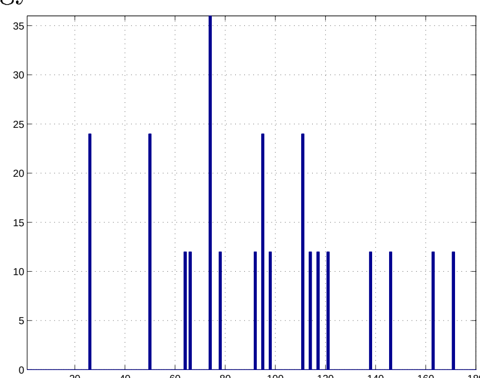
4.10.7 Two Rings

Once again a two-ring equilibrium case [14], the $(12, 12^s)$ configuration is not a stable equilibrium. The energy is much higher and the radial distribution function, while not symmetric, is quite regular.

Monte Carlo State Energy: 52.69529432435961



Latitude-Longitude Map
Figure 4.10.7a

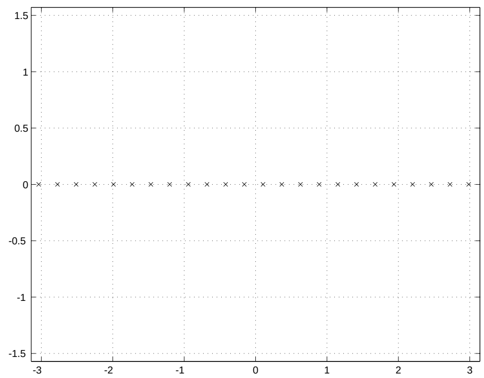


Radial Distribution Function
Figure 4.10.7b

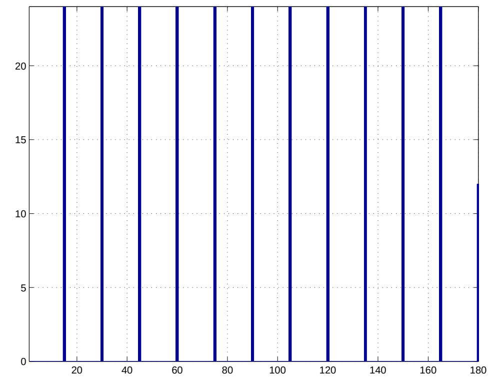
4.10.8 One Ring

The (24) configuration, 24 vortices uniformly spaced on the equator, is the highest energy state found by the methods in this paper. The regular uniformity of the radial distribution function is seen as in sections 4.1.4, 4.3.5, 4.5.5, 4.8.8, 4.9.7.

Monte Carlo State Energy: 115.0353299066417



Latitude-Longitude Map
Figure 4.10.8a



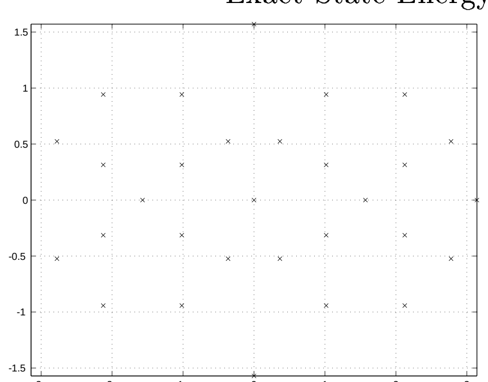
Radial Distribution Function
Figure 4.10.8b

4.11 Thirty Vortices

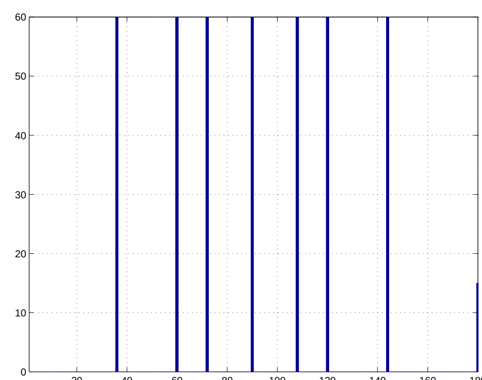
4.11.1 Icosidodecahedron

The icosidodecahedron is not one of the Platonic solids, but is one of the Archimedean solids [18]. It is dual to the dodecahedron, created by bisecting each of the edges in the dodecahedron and connecting those points of bisection to one another [4].

Exact State Energy: 76.65276522704850



Latitude-Longitude Map
Figure 4.11.1a

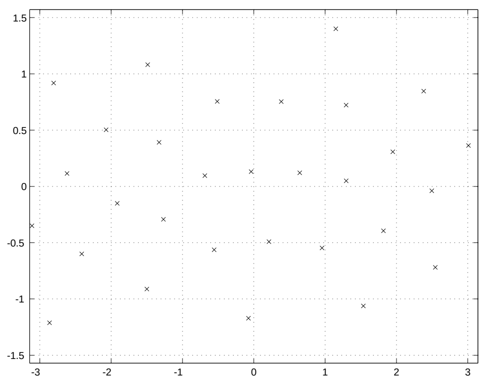


Radial Distribution Function
Figure 4.11.1b

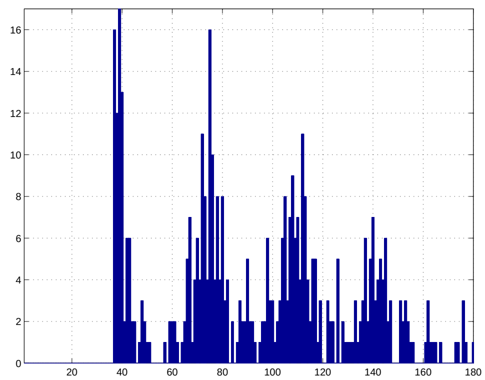
4.11.2 Free Particles

The ground state has an energy distinctly lower than that of the icosidodecahedron (which is again an Archimedean solid [18]). This configuration, which under rotation becomes $(3, 3^s, 6^s, 3^s, 6^s, 3^s, 3^s, 3^s)$, may not be an obvious polyhedron. As was done for several sections (4.10.1, 4.10.2, 4.10.3, 4.10.4) for twenty-four particles, this run was done for 10,000 sweeps.

Monte Carlo State Energy: 75.34072839080393



Latitude-Longitude Map
Figure 4.11.2a

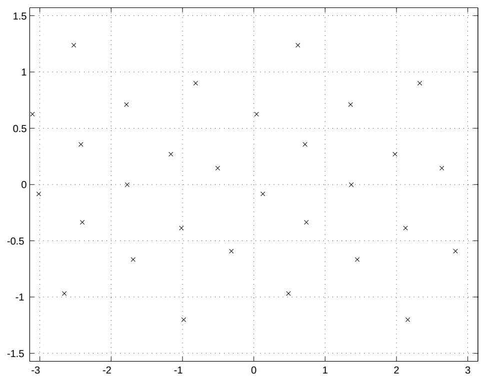


Radial Distribution Function
Figure 4.11.2b

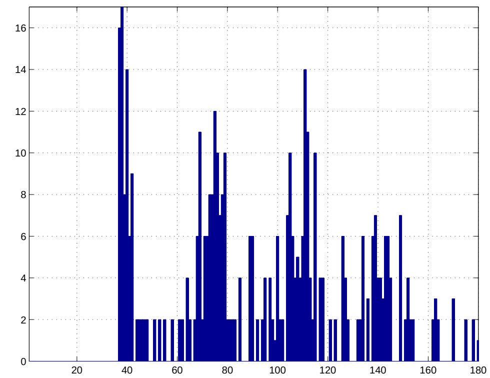
4.11.3 Fifteen Rings

Fifteen rings of two vortices each again matches the ground state found by the free particles. As has been seen many times the state of rings of two vortices each is congruent to the free particle case (as seen in sections 4.1.3, 4.3.3, 4.5.3, et cetera). This was, as above, done with 10,000 sweeps.

Monte Carlo State Energy: 75.34201456372053



Latitude-Longitude Map
Figure 4.11.3a

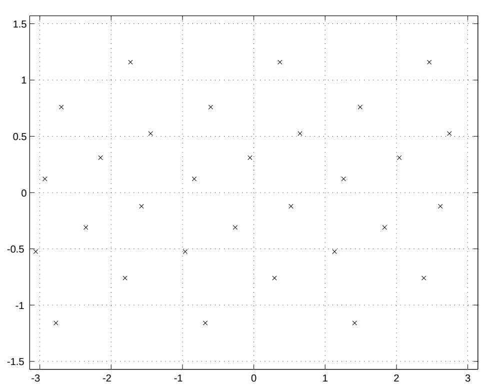


Radial Distribution Function
Figure 4.11.3b

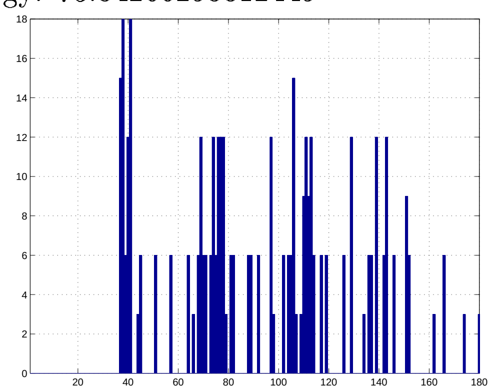
4.11.4 Ten Rings

Ten rings with three vortices to each latitudinal ring again reaches the configuration of the ground state $(3, 3^s, 6^s, 3^s, 6^s, 3^s, 3^s, 3^s)$. Its radial distribution function is sharper than for the cases with more independently moving rings or vortices. As with sections 4.11.2 and 4.11.3 the results are from 10,000 sweeps.

Monte Carlo State Energy: 75.34266293812449



Latitude-Longitude Map
Figure 4.11.4a

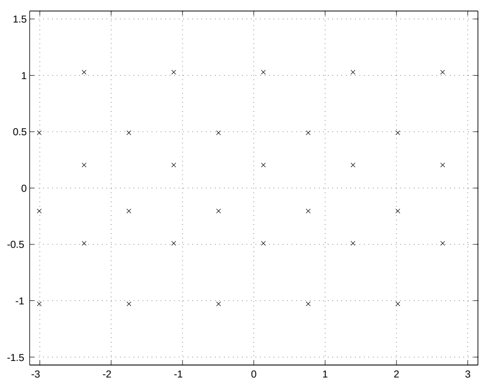


Radial Distribution Function
Figure 4.11.4b

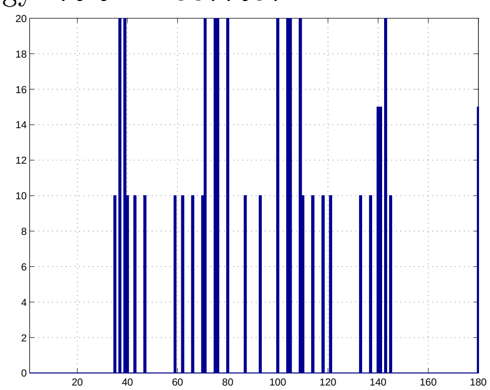
4.11.5 Six Rings

The configuration $(5, 5^s, 5^s, 5^s, 5^s, 5^s)$ is not unexpected and the energy of the state is barely higher than the ground state. These results are from 10,000 sweeps.

Monte Carlo State Energy: 75.52411937759742



Latitude-Longitude Map
Figure 4.11.5a

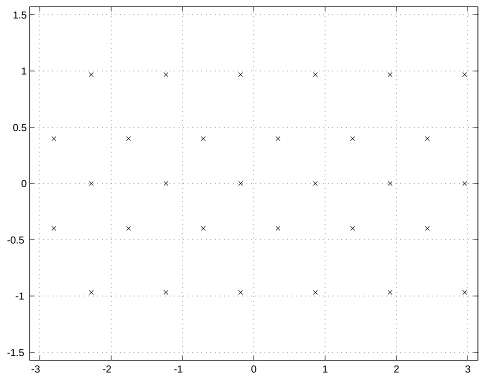


Radial Distribution Function
Figure 4.11.5b

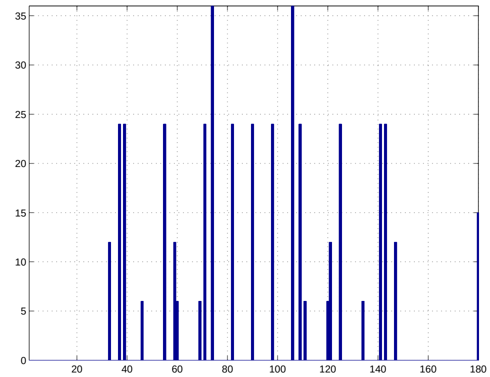
4.11.6 Five Rings

The energy for this state, the configuration $(6, 6^s, 6^s, 6^s, 6^s)$, is much closer that of the icosidodecahedron. As has been observed (section 4.10.6) one ring settled on the equator, and two settled in the northern and the southern hemispheres. In this case the alignment of longitude of alternating rings is very good. These results were arrived at following only 1,000 sweeps, rather than the 10,000 used in the sections above.

Monte Carlo State Energy: 76.07009295197271



Latitude-Longitude Map
Figure 4.11.6a

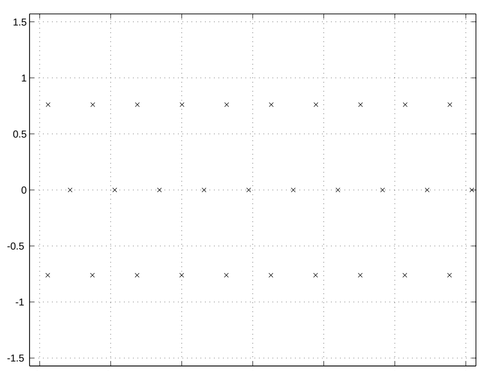


Radial Distribution Function
Figure 4.11.6b

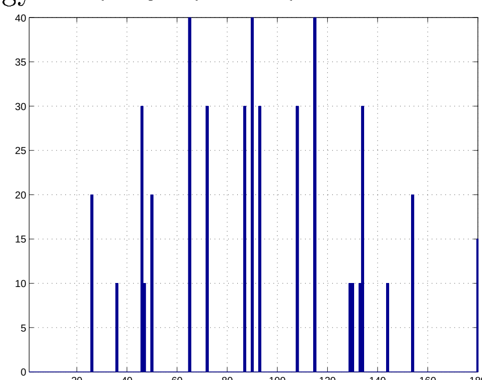
4.11.7 Three Rings

The $(10, 10^s, 10^s)$ case has a slightly higher energy and again the system settles into one equatorial ring, one northern, and one southern, as in section 4.10.6. The radial distribution function is highly symmetric around $\frac{\pi}{2}$ radians.

Monte Carlo State Energy: 80.72252271226274



Latitude-Longitude Map
Figure 4.11.7a

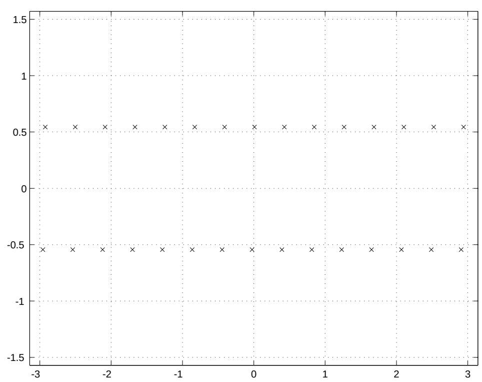


Radial Distribution Function
Figure 4.11.7b

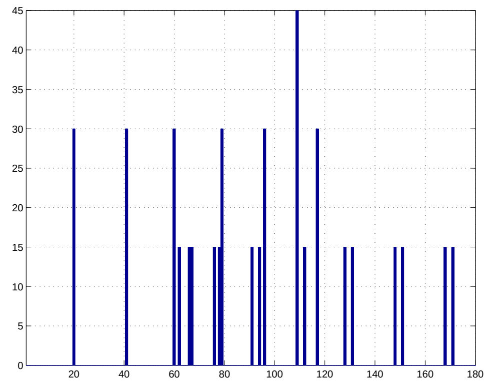
4.11.8 Two Rings

The $(15, 15^s)$ equilibrium state – the third-highest energy found for thirty particles – has, as in section 4.10.7, a very regular radial distribution function. The latitudinal separation between these two rings has been decreasing steadily as more vortices are added to the system, but they do not (in these investigations) settle below a latitudinal separation of one full radian.

Monte Carlo State Energy: 98.05769294869511



Latitude-Longitude Map
Figure 4.11.8a

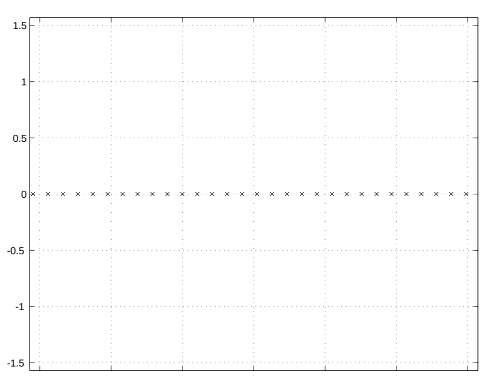


Radial Distribution Function
Figure 4.11.8b

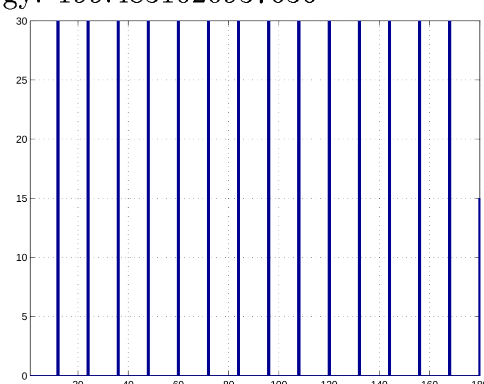
4.11.9 One Ring

As expected from previous sections the case of thirty vortices on a single line of latitude is the highest energy state studied here. The familiar regular radial distribution function is again exhibited.

Monte Carlo State Energy: 199.4831020937636



Latitude-Longitude Map
Figure 4.11.9a



Radial Distribution Function
Figure 4.11.9b

5 Observations

The vertices of the tetrahedron are clearly the coordinates into which vortices settle for the case of $n = 4$ vortices in the Monte Carlo system. This is a configuration straightforward enough that one may see by inspection the $n = 4$ solution can be rotated to derive the vertices of the tetrahedron. The identical radial distribution functions for both configurations serve as confirmation of this identity. Furthermore, the energy of uniform vortices (of strength 1) placed at the corners of the tetrahedron is -1.72609243471069, compared to the energy of the final configuration from the Monte Carlo simulation of -1.72609240.

Similarly the vertices of the octahedron and the configuration of vortices for the case in which $n = 6$ are congruent configurations. The radial distribution function of the octahedron's vertices and that of the simulation's final state are identical. The energy of uniform vortices (of strength 1) placed at the vertices of the octahedron is -2.07944154167984 and that of the final Monte Carlo configuration is -2.07944153.

The other Platonic solid whose energy is matched by the Monte Carlo algorithm is the icosahedron, with $n = 8$ vortices.

It is notable that the vertices of a regular polyhedron can be expected, from the high degree of symmetry each vertex enjoys, to represent an equilibrium configuration for the problem of vortices on the surface of the sphere. That there are therefore generally multiple equilibrium configurations is unsurprising, given that multiple equilibria can be found by constraining some or all of the vortices in a system to latitudinal rings.

The Monte Carlo approach does suggest a method by which one can in essence experimentally test the stability of an equilibrium configuration such as the vertices of a regular polyhedron. As each sweep consists of a number of attempted moves of a given vertex – accepted or rejected based on whether the proposed move decreases or increases the energy of the system – one would expect that if the state begins with a stable equilibrium, any sufficiently small change (assuming the inverse temperature β is sufficiently high) will be rejected as any such change would require the system energy to increase. A configuration that remains impervious to a long Monte Carlo run at very high inverse temperatures may be considered numerically very likely to be a stable equilibrium.

The table below outlines the energies of the regular configurations of vertices as outlined in Coxeter [4] and in Hart [7] if one places vortices of

Figure 2: Energies of Several Polyhedron Configurations

Configuration	N	Exact Energy	Monte Carlo Energy
Platonic Solids			
Tetrahedron	4	-1.72609243471069	-1.72609240
Cube	8	-1.35919229436318	-1.44791449
Octahedron	6	-2.07944154167984	-2.07944143
Icosahedron	12	2.53542345606662	2.53542351
Dodecahedron	20	34.17896965381927	23.67571745
Archimedean Solids			
Cuboctahedron	12	2.74548665548307	2.53542351
Truncated Tetrahedron	12	4.45017794027800	2.53542351
Truncated Cube	24	47.85410832300643	40.88652830
Truncated Octahedron	24	43.16581437199417	40.88652830
Rhombicuboctahedron	24	41.10242806477061	40.88652830
Snub Cube	24	40.99574859752772	40.88652830
Icosidodecahedron	30	76.65276522704850	75.34072839
Miscellaneous			
Double Tetrahedron	5	-1.90954250488417	-1.90954243
Square Antiprism	8	(unavailable)	-1.44791449

Figure 3: The exact energies of several polyhedra compared to the minimum energy found by a Monte Carlo algorithm using as many vortices as the polyhedra have vertices.

uniform strength (1) at each vertex versus the energies found by the Monte Carlo algorithm if one places a number of vortices (of uniform strength) equal to that polyhedron's number of vertices on the unit sphere. In each case the Monte Carlo-generated energy either matches (within an experimental error) the energy of the corresponding polyhedron, as it does for three of the five Platonic Solids, or is below the polyhedron energy.

This Monte Carlo method for finding the energy minimum has an advantage over other methods – such as a steepest-descent search – which are intrinsic to the probabilistic methods used in it. The most significant of those advantages is that the Monte Carlo algorithm is less vulnerable to becoming trapped in a local energy minimum for a system. A steepest descent sort of algorithm can become trapped when the gradient is locally near zero. The Monte Carlo algorithm can explore a region of phase space around even a local minimum.

6 Conclusions

The traditional use of a Monte Carlo algorithm to find statistical equilibria is well documented. It is however also a powerful tool to find dynamic equilibria, combining an ease of implementation with rapid solutions. These results, found from considering free particles and particles constrained in space, suggest a fascinating set of analytical conclusions relating straightforward and previously unconsidered geometrical properties to the existence of dynamic equilibria. While these results have been principally numerical in character their agreement with known analytical results [4] [14] and their extension to previously unidentified but verifiable results provide confidence that the Monte Carlo algorithm is a powerful tool to extend the analytic understanding of N-body problems in several directions [3].

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