Notes on Incomplete One-period Markets and Fundamental Theorem of Asset Pricing

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Let \((M_{m \times n}, S_0, P)\) be a one-period market where \(n \geq m\), that is incomplete, i.e., \(\operatorname{rank}(M) = r < m\). Let \(V \in \mathbb{R} = \text{Range}(M) = \text{colspan}(M)\), that is, it is hedgeable. So there is a hedge \(X \in \mathbb{R}^n\) such that

\[(*) \quad MX = V.\] (1)

Since

\[N + r' = n,\] (2)
\[N' + r = m\] (3)

where \(r = \operatorname{rank}(M) = r' = \operatorname{rank}(M')\), \(N = \dim \text{Nullspace}(M)\), and \(N' = \dim \text{Nullspace}(M')\), we get

\[N = n - r' \geq N' = m - r > 0.\] (4)

Given that \(X\) is a hedge for \(V\), then \(X' = X + k\) also solves (*) for any \(k \in \text{Nullspace}(M)\). Thus,

**Proposition 1** Incomplete one-period markets have uncountably many hedges \(X(V)\) for each hedgeable \(V\).

Consider the Replication method of pricing contingency claims \(V\) including call and put options: for a hedgeable \(V \in \text{Range}(M)\), the Replication price of \(V\) is \(V_0 = X' S_0\) where \(X\) is a hedge for \(V\). Since \(X' = X + k\) is also a hedge for \(V\), \(V_0' = S_0' X' = S_0'(X + k) = S_0' X + S_0' k = V_0 + S_0' k\). Thus,

**Lemma 2** For an incomplete one-period market, the Replication price of hedgeable claim \(V\) is well-defined (agrees for all hedges \(X' \) of \(V\)) if and only if \(\text{Nullspace}(M) \perp S_0\).

On the other hand, consider the second method of pricing claims, based on a Risk-Neutral measure \(q \in \mathbb{R}^m\) based on the dual problem

\[(**) \quad M' q = (1 + r) S_0\] (5)
\[q > 0, \text{ i.e., } 1 > q_j > 0.\] (6)

Given a RN measure \(q\), the RN price of a claim \(V \in \mathbb{R}^m\) is \(V_0 = \frac{1}{1 + r} E_q[V]\) where \(r\) is the interest rate. Since \(N' = m - r > 0\), for each RN \(q\) of the incomplete one-period market, there are uncountably many solutions \(q' = q + k'\) where \(k' \in \text{Nullspace}(M')\). Thus, for \(||k'||\) small enough, \(q'\) is another RN measure. We have proved:
Lemma 3 If an incomplete one-period market has a RN $q$, then $q' = q + k'$ where $k' \in \text{Nullspace}(M')$ is another RN measure provided $||k'||$ is small enough.

Lemma 4 For a incomplete one-period market with a RN $q$, the RN price of a claim $V \in \mathbb{R}^m$, $V'_0 = \frac{1}{1+r} E_q[V]$ is well-defined if and only if $\text{Nullspace}(M') \perp V$.

Proof. $V'_0 = \frac{1}{1+r} E_q[V] = \frac{1}{1+r} (q + k')^t V = \frac{1}{1+r} q^t V + \frac{1}{1+r} (k')^t V = V_0$ if and only if $(k')^t V = 0$ for all $k' \in \text{Nullspace}(M')$. □

Since

$$\text{Null}(M') \oplus \text{Range}(M) = \mathbb{R}^m \quad (7)$$

$$\text{Null}(M) \oplus \text{Range}(M') = \mathbb{R}^n \quad (8)$$

we have $\text{Nullspace}(M) \perp S_0$ if and only if $S_0 \in \text{Range}(M)$. And $S_0 \in \text{Range}(M')$ if and only if $(\ast \ast)$ has a solution $q$ (which need not be a RN measure). We have proved:

Proposition 5 For an incomplete one-period market, the Replication price of hedgeable claim $V$ is well-defined (agrees for all hedges $X'$ of $V$) if and only if the market has a solution $q$ to the dual problem $(\ast \ast)$.

By $\text{Null}(M') \oplus \text{Range}(M) = \mathbb{R}^m$, we have $\text{Nullspace}(M') \perp V$ if and only if $V \in \text{Range}(M)$. Thus,

Lemma 6 For a incomplete one-period market with a RN $q$, the RN price of a claim $V \in \mathbb{R}^m$, $V'_0 = \frac{1}{1+r} E_q[V]$ is well-defined if and only if $V$ is hedgeable.

Combining the previous proposition and lemma, and noting that

$$V_0 = X^t S_0 = \frac{1}{1+r} X^t M' q = \frac{1}{1+r} (M X)^t q = \frac{1}{1+r} E_q[V], \quad (9)$$

we obtain the proof of the following:

Theorem 7 In an incomplete one-period market with a RN $q$, every hedgeable claim $V$ has a unique replication price $V_0 = X^t S_0$ which agrees with its RN price $V'_0 = \frac{1}{1+r} E_q[V]$.

Next, we relate these results to the concept of arbitrage-free (AF) one-period markets.

Definition 8 An arbitrage is a hedge $X$ for a claim $V \in \mathbb{R}^m$ such that

(i) $V_0 = X^t S_0 = 0$, either (ii) $V \geq 0$, and $V_j > 0$ for some $j = 1, \ldots, m$

or (ii)' $V \leq 0$, and $V_i < 0$ for some $i = 1, \ldots, m$

Definition 9 A one-period market is AF if there are no arbitrages in it.
Theorem 10 If a one-period market has a RN $q$, then it is AF.

Proof. By previous theorem, let $X(V)$ be an arbitrage. Then, since $q > 0$, and wlog we can take $V \geq 0$, with $V_j > 0$, we get

$$0 = X^tS_0 = \frac{1}{1+r}E_q[V] > 0$$

(10)

which is a contradiction. QED. ■

Theorem 11 (Converse) If a one-period complete market is AF, then it has a RN $q$.

Proof. Let $X(V)$ be any hedge such that $0 = X^tS_0$, then AF implies that $V = MX$ equals 0 or has mixed signed entries. That is, $U = \{V = MX \mid 0 = X^tS_0\}$ is a vector subspace of $R^m$ that intersects the union $K \cup -K$ only at the origin 0, where $K$ is the closure of the first octant. Then there is a $0 \neq q \in R^m$ such that $q \cdot V = 0$ for all $V \in U$, and $q \cdot V > 0$ (resp. $< 0$) if $V \in K \setminus \{0\}$ (resp. $-K \setminus \{0\}$) by the Separating Plane Theorem (Hahn-Banach).

$U$ separates the closed convex sets $K$ and $-K$. It can be normalized so that $\sum_{j=1}^m q_j = 1$. Also such a $q$ must be in the interior of $K$; otherwise if $q \in \partial K$, then there is some $V \in \partial K$ for which $q \cdot V = 0$. Lastly, $q^tV = q^tMX = 0$ for all $V$ in $U$, implies that $q^tMX = S_0^tX$ for all $X \in W = \{X \in R^n \mid X \cdot S_0 = 0\}$ such that $V$ is in $U$. This means that $M^tq \parallel S_0$ because $\text{co dim } W = 1$. QED. ■