Lecture 1. The Basics

risk - involves an unknown outcome, but a known probability distribution

uncertainty - involves an unknown outcome without a known probability distribution

- Our objective is to manage the uncertainty by balancing risk and return

Gamble 1. You are now Banker 1. As a crafty gambler, you offer this bet to willing players:

“The player pays Banker 1 $1. Then, Banker 1 flips a coin. If the coin lands on heads, then the player receives $4 from Banker 1. Otherwise, the player receives nothing. The coin is loaded such that it lands on heads 30% of the time and tails 70% of the time.”

random variable - a variable that can take on a number of different values, each with an associated probability

expected value (of a random variable) - the weighted average of all possible values that this variable can take on

\[ E[S_1] = \sum_{j=1}^{m} p_j s_1^j \]

- For Bank 1:

\[ E[S_1] = (.30)($4) + (.70)($0) = $1.20 \]

(In this scenario, the player expects to win $1.20, minus the $1 to play = $0.20)

As Banker 1, what do you need to do to avoid bankruptcy?

- First, let’s look at what you, as Banker 1, have access to.
The Market

- Gamble 1 plays the role of one stock in our market. Beginning with a fixed stock price $S_0$, the stock will have the value of either with random probability after one unit of time in the future (the single period binomial pricing model).

\[ \begin{align*}
S_0 & \quad \Rightarrow S_{1}^A \\
S_0 & \quad \Rightarrow S_{1}^B
\end{align*} \]

(the subscript “0” indicates that $S_0$ is the initial stock value, while the “1” denotes the time at the end of one period)

The Market offers multiple stocks. Banker 2 offers a similar gamble, but when his players flip a heads, they receive $5.

\[ \begin{align*}
1 & \quad \Rightarrow A \quad 5 \\
1 & \quad \Rightarrow B \quad 0
\end{align*} \]

A third gamble:

\[ \begin{align*}
1 & \quad \Rightarrow A \quad 6 \\
1 & \quad \Rightarrow B \quad 1
\end{align*} \]

with the same coin.

Here’s a snapshot of the market:
Reducing Risk

- Now that we have our market, we return to the original question: how would Banker 1 avoid bankruptcy?
- Recall that, as Banker 1, you have two possible outcomes at the end of the period.
  - **outcome 1**: the coin lands on heads, so you must pay the player $S_{1A} = $4. But, you keep the original $S_0 = $1 to play. Your net loss is $3.
  - **outcome 2**: the coin lands on tails, so the player receives $S_{1B} = $0, and you keep the original $S_0 = $1 to play. Your net gain is $1.
- How much money should Banker 1 have on hand before he offers such a bet? The most conservative amount needed is $3 per player to account for the possibility of every coin landing on heads. This might be feasible for a $3 deal, but in the real world, accounting perfectly for the worst-case scenario is not always practical. When the value of the stock is worth hundreds of dollars instead of $3 or $4, Banker 1 might have to borrow money from the bank to cover his costs, and a perfectly conservative estimate will cause him to pay more in interest in the long run. Alternatively, Banker 1 might set aside $0.20, the expected return to the player minus the $1 you keep at time 0.

-OR-

Banker 1 can take advantage of his knowledge of the market, and **hedge** his portfolio—that is, solve a system of equations to tell Banker 1 which other stocks in the market to invest in to offset potential losses from Gamble 1.

**Question:** What combination of stocks in our market is equivalent to our Gamble 1? Is such a combination even possible?

- Let:
  \[ x = \text{the number of stocks of } S^1 \text{ to be purchased} \]
y = the number of stocks of $S^2$ to be purchased

\[
\begin{align*}
xs_1^A + ys_1^{2A} &= s_1^A \\
x s_1^B + y s_1^{2B} &= s_1^B
\end{align*}
\]

Here, we have:

\[
\begin{align*}
5x + 6y &= 4 \\
0x + 1y &= 1
\end{align*}
\]

In matrix form,

\[
Mx = s_1
\]

\[
\begin{bmatrix} s_1^A & s_1^{2A} \\ s_1^B & s_1^{2B} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_1^A \\ s_1^B \end{bmatrix}
\]

\[
\begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}
\]

If \( \text{det}(M) \neq 0 \), we can solve for \( x \) and \( y \) (hedging).

\[
\text{det} \left( \begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix} \right) = 5
\]

- Solving, we have that \( y = 1 \) and \( x = -0.4 \). Banker 1, then, should sell (or “short”) \( 0.4N_{G1} \)
  and buy \( 1N_{G2} \), where \( N = \) the number of each respective Gamble he will offer. Let’s compare the returns.

**Without hedging**

Returns:

\[
R^A = \frac{s_1^A - s_0}{s_0} = \frac{4-1}{1} = 3.
\]

- Without hedging, the player will make \$3, so Banker 1 will lose \$3 in the up-market.

\[
R^B = \frac{s_1^B - s_0}{s_0} = \frac{0-1}{1} = -1 \quad \text{(banker gain)}
\]

- Without hedging, the player will lose \$1 in the down-market.

**With hedging**

- The initial cost of each stock in our market was \$1; that is, \( S_0 = S_0^1 = S_0^2 \).
Stock 1: \( S_0^1 x = (\$1)(-0.4) = -0.40 \)

Stock 2: \( S_0^2 y = (\$1)(1) = 1.00 \)

\[ \text{\$0.60 gain} \]

- By adjusting his portfolio, Banker 1 has \$0.60 per Gamble to work with.

  Considering that the expected value of \( S_0 \), \( E[S_0] = -0.20 \), the Banker is good to go!

- In Lecture 2, we will try another example of this with realistic stock prices. First, let’s review (for the first time) some statistics terms:

**Statistics**

**expected value:** \( E[X] = \sum_{j=1}^{m} p_j X_j \), where \( \sum_{j=1}^{m} p_j = 1, p_j \geq 0 \).

The \( p_j \)'s, in our case, are the probabilities of each outcome.

**variance:** \( \text{Var}(X) = E[(X - E[X])^2] \)

This gives a measure of how far the outcomes lie from the expected value.

- Here’s one more stock as an example:

  \[ S_0 = \$1 \]

  \( S_1^A = \$4 \quad 30\% \text{ probability} \)

  \( S_1^B = \$0.50 \quad 70\% \text{ probability} \)

  \( E[S_1] = (0.30)(\$4) + (0.70)(\$0.50) = 1.55 \)

  \( \text{Var}[S_1] = E[(S_1 - E[S_1])^2] = 0.3(4 - 1.55)^2 + 0.7(0.5 - 1.55)^2 = 2.5725. \)
Lecture 2. Worksheet Day- Hedging and Options

- Now we’re going to look at hedging again, this time with more realistic stock prices.

For simplicity, let’s re-use Gamble 1 from the previous lecture.

\[
\text{Gamble 1: } \$1 \quad \text{\rightarrow} \quad \begin{cases} 4 \quad .3 \\ 0 \quad .7 \end{cases}
\]

Observing current stock prices, we can choose a market consisting of two shares, Apple (AAPL) and IBM.

- AAPL: \( S_0 = $670 \quad \rightarrow \quad S_{1A} = $700 \)
- IBM: \( S_0^1 = $190 \quad \rightarrow \quad S_{1B}^1 = $170 \)

Solving, we have that
\[
x \approx -5.1 \text{ shares of AAPL, and } y \approx 18.95 \text{ shares of IBM.}
\]

What is happening financially at time \( t = 0 \)?
- \( x\cdot S_0 = (-5.1)\cdot($670) = -$3400 \)
  This is a short- you borrow shares from the brokerage, sell them, and hope to buy them back later. You will profit if the price goes down.
- \( y\cdot S_0 = (18.95)\cdot($190) = $3600 \)
  This is a long- you buy shares of IBM and profit later if the price goes up.

What is happening financially at time \( t = 1 \)?
(Six months later, where we choose a single period to last 6 months)

Building the market matrix \( M \):
\[
\begin{bmatrix} 700 & 210 \\ 630 & 170 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 400 \\ 0 \end{bmatrix}
\]

Units: \( (\text{dollars per share})(\text{share}) = (\text{dollars}) \)

Note that we are working with 100 units of our Gamble 1 stock in this system of equations. Since this is a linear system, the solutions will scale appropriately (and if you don’t believe me, try it with \([-4, 0]^T\)! Any constant multiple of the vector will work.
We close the books! Let’s see how our 100 bets did.

Scenarios:
(A) heads  30% chance
   The banker pays the better $400 (but keeps the initial $100).
(B) tails  70% chance
   The banker pays the better $0.

Again, we have about $200 to work with as a result of our initial hedging, so our risks are reduced. The banker only has to come up with $100 in the event of a heads-flip!

What’s with the coin?

It seems odd to use a rigged coin for estimating outcomes. However, looking at recent financial records allows us to estimate \( p \), and after a long period of time, the coin-toss model gives a reasonably good prediction of the market. Shreve says that with a sufficient number of periods, the binomial asset-pricing model provides a reasonably good, computationally tractable approximation to continuous-time models. We can also consider the question of what odds must Banker 1 offer to not lose money!

[Worksheet 1]

Puts and Calls- the two types of stock option contracts

For the first part of the semester, we will deal only with European put and call options. Here’s how they work:

- **call**: At time \( t = 0 \), the value of a stock is \( S_0 \) and we set a **strike price**, \( K \). An expiration date is set. At the end of this period, the buyer of the call option has the **right to buy** the stock at the strike price, regardless of the final value of the stock. So, the buyer of the call expects the price of the stock to rise, and profit from **buying the call at a lower price** than its current value.
  - Example: EBAY
    - Price at \( t=0 \): $46.74
    - Strike price(call): $42.00 = K
    - Days of trading: 45 (end date Oct. 12)
    - Cost to buy the call: $5.70

Imagine that these are the possible outcomes (as predicted from recent financial data):

\[
S_0 = 46.74 \quad \rightarrow \quad S_{1A}^* = 52 \quad 40\%
\]

\[
S_0 = 46.74 \quad \rightarrow \quad S_{1B}^* = 43 \quad 60\%
\]
The payoffs for each outcome:
\[ P_1^A = (S_1^A - K) = $10 \]
\[ P_1^B = (S_1^B - K) = $1 \]

BUT we must take into account the cost to buy the call. Regardless, we can use the market matrix \( M \) from Worksheet 1 to hedge, based on the payoffs:

\[
M_{10}^* = \begin{bmatrix} $10 \\ $1 \end{bmatrix}
\]

\( \rightarrow \) (long) IBM: \( x = .48 \), gain of $79.26

(short) AAPL: \( y = -11 \), loss of $73.70.

- **put**: At \( t=0 \), the value of the stock is \( S_0 \) and we set a strike price, \( K \). An expiration date is set. At the end of this period, the buyer of the put option has the right to sell the stock at the strike price, regardless of the final value of the stock. So, the buyer of the put expects the value of the stock to fall and profit by selling the stock at a higher price than its current value.

- **European options** allow the holder to exercise the option for a short period of time before the expiration date, while **American options** can be exercised at any time before the expiration.
Lecture 3. Pricing options, arbitrage and the benefits of hedging

A quick review of single-period 2-state markets:

The market matrix: m×n
- m rows - outcomes at the end of a single period
- n columns - payoffs at the end of the period of the m assets
- asset - any economic resource, like stocks (S) and bonds (B)

\[
\begin{bmatrix}
S_u & B_u \\
S_d & B_d
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
C_u \\
C_d
\end{bmatrix}
\]

\[\begin{bmatrix}
M
\end{bmatrix} \quad \text{“C” = Contingency Claim (C.C.) - the payoff of the bet}
\]

bond - a riskless asset; however, the issuer also owes the holder interest, depending on the terms of the bond.
- If the bond is worth \(B_0\) at the time \(t=0\), then after one period, \(B_u = B_d = B_0(1+r)\), where \(r\) is the interest rate.
- We typically normalize the bond such that \(B_0 = 1\); then our market matrix takes the form:

\[
M = \begin{bmatrix}
S_u (1 + r) \\
S_d (1 + r)
\end{bmatrix}
\]

The hedging problem. Find a replicating portfolio (hedge portfolio \([x \ y]^T\)) such that when you buy \(x\) units of \(S_0\) and \(y\) units of \(B_0\) at \(t=0\), you will hold a portfolio that will generate \(C_u\) with (heads) probability \(p\) and \(C_d\) with (tails) probability \(q\), where \(p + q = 1\), and \(p, q \geq 0\).

\[
\begin{align*}
(B_u = B_d) & \quad \begin{cases}
S_u x + B_u y = C_u \\
S_d x + B_d y = C_d
\end{cases}
\]

Can we find a hedge? If \(\det(M) = S_u B_d - S_d B_u \neq 0\), then all contingency claims \([C_u \ C_d]^T\) can be hedged in this market. We call this a complete market.

Example 1. A call option

Let’s use SPY stock and a bond with an interest rate of .3%. Fix \(p = q = \frac{1}{2}\). A replicating hedge will show us how to price the call.

We have:
AAPL stock and its projections for the end of the period, Oct. 12:

\[ S_0 = 144 \]

\[ S_1^A = 700 \]

\[ S_1^B = 650 \]

Note that the strike price for the AAPL call is set as \( K = S_0 \). At the end of the period (on Oct. 12), the buyer has the **right to buy** the stock at \( $K \). This is in anticipation of the stock increasing in value \( (S_u) \). If the stock decreases in value \( (S_d) \), the buyer will not exercise the contract.

Ignoring the fees to buy the call, the contingency claim is

\[ \vec{c} \cdot \vec{c} = \begin{bmatrix} [S_u - K]^+ \\ [S_d - K]^+ \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \end{bmatrix} \]

\[ \uparrow \]

This denotes the positive value within the brackets, so negative values are set to zero. Financially, this means that the buyer of the call is not exercising his contract.

Solving

\[ [150 \ 1.003] [x] \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} C_u \\ C_d \end{bmatrix} \]

\[ S_0 = 144 \]

\[ S_1^A = 700 \]

\[ S_1^B = 650 \]

\[ x = 10 \text{ units}, \ y = -1465 \text{ units}. \]

**Pricing the call**

\[ C_0 = xS_0 + yB_0 = 1440 - 1465 = -25. \]

This is a **bad market**. The call value is nonsensical! We must find a characteristic of better markets.

Here’s an example of a **better market**:

\[ \begin{bmatrix} 700 \ 1.003 \\ 650 \ 1.003 \end{bmatrix} [x] \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \end{bmatrix} \]
\[ S_0 = 670 \]

When \( \mathbf{M} \) has this form,

\[
\mathbf{M} = \begin{bmatrix} S_u \ (1 + r) \\ S_d \ (1 + r) \end{bmatrix},
\]

we already know that the quality of the market to be complete, and thus always allow us to solve for our contingency claim, is \( \det(\mathbf{M}) \neq 0 \); that is, when \( S_u > S_d \).

Here, \( x = .6, \ y = -388.8, \) and \( C_0 = xS_0 + yB_0 = (.6)(670) - 388.8 = 13.20 \).

**Arbitrage**

Examining the bad market:

\[
\begin{bmatrix} 150 & 1.003 \\ 147 & 1.003 \end{bmatrix}
\]

where \( S_0 = $144 \).

At \( t=0 \), we can borrow \$144 to buy one unit of SPY.

In the up-market:

We gain \((150 - 144) - (144)(.003) = 5.60\)

In the down-market:

We gain \((147 - 144) - (144)(.003) = 1.60\)

You can’t lose! The market is bad because, somewhere, someone is guaranteed a “free lunch” regardless of the outcome. The market is not balanced. We say that there is arbitrage.

**arbitrage**—the possibility of a risk-free profit at zero cost. Shreve notes that, “Wealth can be generated from nothing in such a model… Real markets sometimes exhibit arbitrage, but this is necessarily fleeting; as soon as someone discovers it, trading takes place that removes it” (2).

- \( \mathbf{M} \) is arbitrage free if \( S_d < S_0(1+r) < S_u \).

**No Arbitrage**

- Our good market:

\[
\mathbf{M} = \begin{bmatrix} 700 & 1.003 \\ 650 & 1.003 \end{bmatrix}, \ S_0 = $670
\]

\( \mathbf{M} \) has no arbitrage, since

\( S_d = 650 < S_0(1+r) = 670*(1.006) = 674.02 < S_u = 700 \).

Since there is no arbitrage, we are guaranteed to be able to price puts.
Using AAPL stock, our contingency claim is now

\[ C.C. = \left[ \begin{array}{c} 0 \\ 20 \end{array} \right] \]

because in the up-market, the buyer of the put would not choose to exercise the contract and sell the stock to the buyer at the strike price when he could sell the stock for the higher price in the market.

Solving the system, we have \( x = -0.4, y = 278.3 \); so

\[ P_0 = xS_0 + yB_0 = (-0.4)(\$670) + \$278.3 = \$10.30. \]

Outcomes:

- In the up-market scenario, you have 1 unit of the put but do not exercise it.
  
  \[ \text{net loss} = (1.006)(\$10.30) = \$10.36 \]

- In the down-market, you have 1 unit of the put and exercise it, so
  
  \[ \text{net gain} = (\$670 - \$650) - (1.006)(\$10.30) = \$9.64 \]

Note: this method of option-pricing is the arbitrage pricing theory approach to the problem. We say that we “replicate” the option by trading in the stock and money markets. The strike price is also called the “no-arbitrage price of the option at time 0”.

**A brief note on terminology:**

- The buyer is short on the underlying asset of a put, but long on the put option itself, because the buyer wants the asset value to decline.

- The seller is long on the asset and short on the put option, because the seller wants the option to become worthless when the stock rises above the strike price.

**The benefits of options:**

**covered call**- for every call bought, you own a share of the same stock. This transaction provides downside protection. These stand in opposition to **naked calls**, in which there is no additional stock or instrument.

- Example: you own one stock of AAPL and are writing one unit of the same AAPL stock call; \( C_0 = \$13.30 \).

\[ \begin{array}{c}
\text{AAPL} \\
S_0 = \$670
\end{array}\]

\[ S_1^A = \$700 \]

\[ S_1^B = \$650 \]
With the protection of writing the call, we have minimized our gains but also our losses. (This idea of shortening the distance between gains and losses gives us the image of the “hedge” between the two.) In the up-market, we lose the $30 gain when the buyer of the call exercises the contract, but we retain the call price. In the down-market we minimize our loss with the money retained from the call price.

- We can also minimize our risk by using puts. To hedge, say we are long by one share of AAPL and we buy one unit of put.
  
  Price of put $P_0 = $10.30

<table>
<thead>
<tr>
<th></th>
<th>gains from stock</th>
<th>protection</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>up-market</td>
<td>$30</td>
<td>-$30 + $13.30</td>
<td>$13.30</td>
</tr>
<tr>
<td>down-market</td>
<td>-$20</td>
<td>$13.30</td>
<td>$6.70</td>
</tr>
</tbody>
</table>

Again, we have successfully hedged our gains and losses.

- At expiry $T$, it is easy to price derivatives:
  
  $C_T = [S_T - K]_+$
  
  $P_T = [K - S_T]_+$
Ultimately, we will look at derivatives as a function of several other variables:

Call: $C_t = f(S_0, S_t, \text{vol}, r, K), \ t \in [0,T], \text{ and}$

Put: $P_t = f(S_0, S_t, \text{vol}, r, K), \ t \in [0,T], \text{ where “vol” is volatility.}$
Lecture 4. Worksheet Day - Hedging and Insurance

- In our two-state market with two assets,
  \[ M = \begin{bmatrix} S_u (1 + r) \\ S_d (1 + r) \end{bmatrix}, \]
  and our real-world probabilities take the form \((p, q = 1 - p)\), where \(p\) and \(q\) are positive and \(p + q = 1\).

- **Definition 1.** \(M\) is **complete** is for every contingency claim \(\overline{C.}\overline{C.} = [V_u V_d]^T\), there is a replicating portfolio (hedge).

  - \(M^{[x]}_{[y]} = \begin{bmatrix} V_u \\ V_d \end{bmatrix}\)

  (Note: we cannot have a complete market if there is arbitrage. However, a market with no arbitrage is not necessarily a complete market. We will see this later.)

Worksheet notes:

- The four parties involved in the transactions:
  - Seller
  - Brokerage - facilitates the buying and selling of securities (e.g., stocks and bonds) between a buyer and seller. A “full service” brokerage is also responsible for researching the markets to provide recommendations to pension fund/portfolio managers.
  - Bank - gives out loans and collects interest.
  - Buyer

- Keep in mind: Who benefits the most from each transaction?

**Insurance**

- Our original contingency claim vector has the form:
  \[ V_0 \rightarrow \begin{bmatrix} V_u \\ V_d \end{bmatrix} = \overline{C.}\overline{C.}. \]

When we buy insurance, we are paying a premium of \(C\) that pays back compensation on the downside.

\[
\begin{bmatrix} V_u' \\ V_d' \end{bmatrix} = \begin{bmatrix} V_u - C \text{ (premium)} \\ V_d - C \text{ (premium) + Compensation} \end{bmatrix}
\]

The returns vector can be written simply as,

\[
\begin{bmatrix} 0 \\ \text{Compensation} \end{bmatrix}.
\]
Compensation is a function of two values:

\[ \text{Comp.} = C*b, \]

where \(C\) is the premium paid at \(t=0\).

In a good market, we can calculate the \$C premium by solving our usual system of equations for hedging, with the compensation vector in its place:

\[
\begin{bmatrix}
V_u (1 + r) \\
V_d (1 + r)
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
0 \\
bC
\end{bmatrix}
\]

Again, we say that we have a good market if \(V_d < V_0(1+r) < V_u\).
Lecture 5. Risk-Neutral Probability Vectors and Complete Markets

Single-Period Markets:

- **Definition 1.** A market is an $m \times n$ matrix, consisting of $m$ states at time 1 and $n$ assets, $(n-1)$ of which are risky assets.

  **Properties:**
  - The market has this probability vector associated with it:
    \[ Q^T = (q_1, \ldots, q_m), \sum_{j=1}^{m} q_j = 1, q_j \geq 0, \ j = 1, \ldots, m. \]

    A subset of the set of probability vectors is the risk-neutral probability vector, in which $q_j > 0, \ j = 1, \ldots, m.$
  - $M$ also has the no-arbitrage property.

- **Definition 2.** $M$ is arbitrage-free if there are no hedging portfolios \( \{ X^T \in \mathbb{R}^n: X^T = (x_1, \ldots, x_{n-1}, x_n) \} \), (with $x_n$ as our riskless asset) where $M X = V_1$, such that:
  a) $V_0 \equiv \sum_{j=1}^{n} X_j S_0^f = X \cdot S_0 = 0$
  b) $V^* \equiv \frac{1}{(1+r)} V \geq 0$ (read: each component of $V \geq 0$)
  c) $V \neq 0.$ (read: at least one component of $V > 0$)

Let’s look at an example. In the $2 \times 2$ case:

\[
M = \begin{bmatrix} S_1^u (1 + r) \\ S_1^d (1 + r) \end{bmatrix}.
\]

And recall that $M$ has no arbitrage if and only if $S_1^d < S_0(1+r) < S_1^u$. We can show that the conditions of Definition 2 verify this result.

Can you figure out, for a given probability vector $Q$, if $M$ has this $Q$ as a risk-neutral probability vector?

\[
Q = \begin{bmatrix} p \\ 1 - p \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

- A risk-neutral decision-maker considers the certain money equivalent of a loss in a bet to be equal to the probability of the loss times the amount of the loss. The same applies to gains. So, the risk-neutral decision-maker would consider a 20% chance of winning $1 million to be equivalent to winning $200,000. Thus, risks are simply sets of possible loss amounts each with an associated probability.
Using linear algebra and this vectorization of risk, we can write:

$$Q^T M = (1+r)S_0^T$$

$$[q_1 \ q_2] \begin{bmatrix} S_1^u (1 + r) \\ S_1^d (1 + r) \end{bmatrix} = (1+r) [S_0 \ 1]$$

In other words, the risks are given on the left-hand side of the equation by multiplying the probability of an outcome occurring with the outcome, where the outcomes of the risky and riskless assets are expressed in $M$. This risk-neutral value should be equivalent to the initial values of the risky and riskless assets plus the interest rate, since we are interested in the time 1 state.

Writing the system of linear equations, we can solve for $q_1$ and $q_2$ to find the set of risk-neutral probability vector possibilities:

$q_1 = p$, for $q_2 = 1 - p$

$pS_1^u + (1 - p)S_1^d = (1 + r)S_0$

$(1 + r)(p + (1 - p)) = (1 + r) \cdot 1$

$p(S_1^u - S_1^d) = (1 + r)S_0 - S_1^d$

$p = \frac{(1+r)S_0 - S_1^d}{S_1^u - S_1^d} = q_1$

$q_2 = 1 - p = \frac{S_1^u -(1+r)S_0}{S_1^u - S_1^d}$

Based on the form of $q_1$ and $q_2$, can you determine when $M$ has any risk-neutral probability vectors?

- **Theorem 1.** $M_{m \times n}$ has a risk-neutral probability vector $Q$ if and only if it is arbitrage free. (Hint: we can see that this is the case for the 2x2 $M$ if we remember our earlier no-arbitrage condition, $S_1^d < S_0(1+r) < S_1^u$.)

- **Definition 3.** In an arbitrage-free market, solvability of condition (a) in Definition 2 for all $V$ necessitates that $M$ is complete.

- Looking at the form of the equation on condition (a), we can indeed identify the form of our pricing equation from Lecture 3,

$$V_0 = \sum_{j=1}^{n} X_j S_0^j = X \cdot S_0 \quad \text{(Definition 2, condition (a))}$$

versus

$$V_0 = xS_0 + yB_0 \quad \text{(option pricing for 2 state, 2 asset $M$)}$$

- If we order the states of $M$, then according to $M_{m \times n} \mathbf{x}^{1 \times n} = \mathbf{V}^{1 \times n}$, $V \in \mathbf{R}^m$ takes the form:
\[ \mathbf{V}^T = (v_1, \ldots, v_m) \]

Payoff in the best possible state

Payoff in the worst possible state

- Quiz question: What is \( E[V | \mathbf{M}, S_0] \)?
  \[ \sum_{j=1}^{m} v_j q_j = \mathbf{V} \cdot \mathbf{Q} = V_1. \]
  The present value is therefore \( \frac{1}{1+r} V_1 \).

- Comparison of hedging portfolios/investments in the same market:
  \[
  \begin{align*}
  \mathbf{M} \mathbf{X}_1 &= \mathbf{V}_a \\
  \mathbf{M} \mathbf{X}_2 &= \mathbf{V}_b
  \end{align*}
  \]
  Given two portfolios, \( \mathbf{X}_a, \mathbf{X}_b \in \mathbb{R}^n \),
  \( \mathbf{V}_0(\mathbf{X}_a) = \mathbf{V}_0(\mathbf{X}_b) \) does not imply that \( \mathbf{X}_a = \mathbf{X}_b \) in \( \mathbb{R}^2 \).

\[
\mathbf{V}_0(\mathbf{X}_a) = \mathbf{V}_0(\mathbf{X}_b) \iff \mathbf{X}_a \cdot S_0 = \mathbf{X}_b \cdot S_0
\]

Diagrammatically:

- Calculating the future expected payoff of a given portfolio:
  \[
  \mathbf{V}_1(\mathbf{X}) = \mathbf{M} \mathbf{X} = \sum_{j=1}^{n} X_j S_1
  \]
  \[
  E_\mathbf{Q}[\mathbf{V}_1(\mathbf{X})] = \sum_{k=1}^{m} V_k (\omega_1) q_k = \text{future expected payoff}
  \]
Thus, the discounted value should be

$$E_Q[V^*(\omega_1)] = \frac{1}{(1+r)} \sum_{k=1}^{m} V_k(\omega_1)q_k,$$

accounting for the interest rate.

- **Calculating the mean payoff and related values:**
  - $E[\text{payoff}] = pS_1^u + qS_1^d$
  - $E[\text{gain/loss}] = p(S_1^u - S_0) + q(S_1^d - S_0)$
    $= pS_1^u + qS_1^d - S_0$
    $= E[\text{payoff}] - S_0$
    $= (\text{average payoff}) - (\text{the initial price}).$
  - $E[\text{return}] = p\left(\frac{S_1^u - S_0}{S_0}\right) + q\left(\frac{S_1^d - S_0}{S_0}\right)$

- **Calculating the payoff variance:**
  - $E[V_1] = pS_1^u + qS_1^d$
  - $\text{Var}(\text{payoff}) = E[(V_1 - E[V_1])^2]$
    $E[V_1] = pS_1^u + qS_1^d$
    $\text{Var}(\text{payoff}, V_1) = p(S_1^u - V_1)^2 + q(S_1^d - V_1)^2$
    $= p((S_1^u)^2 - 2S_1^uV_1 + (V_1)^2) + q((S_1^d)^2 - 2S_1^dV_1 + (V_1)^2)$
    $= p(S_1^u)^2 + q(S_1^d)^2 + (p + q)(V_1)^2 - 2V_1(pS_1^u + qS_1^d)$
    $= p(S_1^u)^2 + q(S_1^d)^2 + (p + q)(V_1)^2 - 2V_1(V_1)$
    $= p(S_1^u)^2 + q(S_1^d)^2 - (V_1)^2$
    $= E[S_1] - (E[S_1])^2.$

- **Calculating the returns variance:**
  - Where $R_1$ is a random variable,
    $E[R_1] = p\left(\frac{R_1^u - R_0}{R_0}\right) + q\left(\frac{R_1^d - R_0}{R_0}\right)$
    $E[R_1]^2 = p\left(\frac{R_1^u - R_0}{R_0}\right)^2 + q\left(\frac{R_1^d - R_0}{R_0}\right)^2.$

The market in payoff form:

- Where $\omega$ is the market state, the standard market matrix is:

\[
M = \begin{bmatrix}
S_1^1(\omega_1) & S_1^2(\omega_1) & \cdots & S_1^{n-1}(\omega_1) & (1 + r) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
S_1^1(\omega_m) & S_1^2(\omega_m) & \cdots & S_1^{n-1}(\omega_m) & (1 + r)
\end{bmatrix}
\]

Each column denotes a single random variable.

So, in terms of payoffs, we have:

\[
M^* = \frac{1}{(1+r)}M = \begin{bmatrix}
S_1^1(\omega_1) & S_1^2(\omega_1) & \cdots & S_1^{n-1}(\omega_1) & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
S_1^1(\omega_m) & S_1^2(\omega_m) & \cdots & S_1^{n-1}(\omega_m) & 1
\end{bmatrix}
\]

where $S_1^1(\omega_1) \equiv (is
defined
to\bequal\to)\frac{1}{(1+r)}S_1^1(\omega_1)$, etc.

We simply re-examine the market outcomes before the effects of interest.

The market in (discounted) gain/loss form:

- $M^*_{\text{gains/loss}} = \begin{bmatrix}
\Delta^*S_1^1(\omega_1) & \Delta^*S_1^2(\omega_1) & \cdots & \Delta^*S_1^{n-1}(\omega_1) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta^*S_1^1(\omega_m) & \Delta^*S_1^2(\omega_m) & \cdots & \Delta^*S_1^{n-1}(\omega_m) & 0
\end{bmatrix}
$

where $\Delta^*S_1^1(\omega_1) \equiv \frac{1}{(1+r)}S_1^1(\omega_1) - S_0^1(\omega_1)$.

This is known as the discounted gains/loss form. We can express each column as a sum of vectors, as follows:

\[
\Delta^*S^1 = \frac{1}{(1+r)}S^1 - S_0^1I
\]

Trading

- Now that we have a more complete description of the market in various forms, we can revisit replicating portfolios.
  - Examine this equation: $MX = V$.

In a two-state, two asset market, $M$ is a $2 \times 2$ matrix. $V$ is a vector with length $m$.

Thus, we define $W \in \mathbb{R}^2$ to be a subspace in $\mathbb{R}^2$ of the payoffs. That is,

\[
W = \{V_i^* \in \mathbb{R}^m \mid V_0 = \sum_{j=1}^{n} X_jS_0^j = 0\}
\]
In $\mathbb{R}^2$, 

$$W = \{ V_1* \in \mathbb{R}^2 \mid xS_0^1 + y^*1 = 0 \}.$$ 

$W$ is the set of contingency claims with the extra caveat that $V_0 = 0$. Note that we must be given $S_0$ in addition to the market matrix $M$ to have a complete picture of $W$. Note, too, that in the two-state, two-asset case, the constraint on $V_0$ confines $W$ to a line in the $\mathbb{R}^2$–plane which goes through the origin (a ray):

- This picture can help us understand more about replicating portfolios and risk-neutral probability vectors. Let us define arbitrage in a more convenient way:
  - **Arbitrage** - a payoff whose initial value is 0 and has positive elements after $t=0$:
    - a) $V_0 = 0$,
    - b) $V_i* \geq 0; V* \neq 0$.
  - If our market is arbitrage-free, then the only way to avoid an arbitrage is by satisfying this condition:
    $$W \cap A = \emptyset,$$
    as shown:
Note that $W_1$ is consistent with our constraints for an arbitrage-free market while $W_2$ is not. That is, $W_1 \cap \Lambda = \emptyset$ because the $W_1$-line does not intersect the first quadrant; only the point $(0,0)$ on its boundary, which is not included in $\Lambda$ by part (b) of the definition, $V_i^* \geq 0$; $V^* \neq 0$.

A quadrant is equivalent to an “octant” in $\mathbb{R}^2$.

To add to the picture, consider another subset of the plane:

$$P^\perp = \{ Q \in \mathbb{R}^2 \mid \sum_{j=1}^{2} q_j = q_1 + q_2 = 1; q_1 > 0, q_2 > 0 \},$$

giving us that $Q$ is a probability vector.
We have displayed $W^\perp$, the orthogonal subspace to $W$. Geometrically, we can figure out our risk-neutral probability vector. If $W$ gives us our subspace of payoffs, then the risk-neutral probability vector should lie at the intersection of $W^\perp$ with $P^\perp$, the subspace of possible probability vectors, as labeled. This can be shown by the Hahn-Banach Theorem, even in higher dimensions. Thus, we have this result:

$(M, S_0)$ is arbitrage-free if and only if:

- it has a risk-neutral probability vector $Q$,
- $W \cap \mathcal{A} = \emptyset$, and
- $W^\perp \cap P^\perp \neq \emptyset$.

We can derive the same results from linear algebra. The adjoint linear system to the forward linear system:

$M X = V$

In standard (undiscounted) payoff form we have:

$Q^T M = (1+r)S_0^T$
We can also look at this system using the discounted payoff and discounted gains/loss forms of our market matrix:

\[ Q^T M^* = S_0^T \]

\[ Q^T M^*_\text{disc. gains/loss} = 0, \text{ or,} \]

\[
(q_1, \ldots, q_m) \begin{bmatrix}
\Delta^*S^1_1(\omega_1) & \Delta^*S^2_1(\omega_1) & \cdots & \Delta^*S^n_1(\omega_1) \\
\vdots & \ddots & \ddots & \vdots \\
\Delta^*S^1_m(\omega_m) & \Delta^*S^2_m(\omega_m) & \cdots & \Delta^*S^n_m(\omega_m)
\end{bmatrix} = \vec{0}.
\]

\( M^\text{disc. gains/loss} \) is an \( m \times (n-1) \) matrix, or alternatively can be thought of as an \( m \times n \) matrix in which the last column is all zero’s. Expressing this as a system of equations, we have:

\[
\begin{cases}
q_1\Delta^*S^1_1(\omega_1) + \cdots + q_m\Delta^*S^1_m(\omega_m) = 0 \\
q_1\Delta^*S^2_1(\omega_1) + \cdots + q_m\Delta^*S^2_m(\omega_m) = 0 \\
\vdots \\
q_1\Delta^*S^n_1(\omega_1) + \cdots + q_m\Delta^*S^n_m(\omega_m) = 0
\end{cases}
\]

In a two-state, two asset matrix:

\[ q_1\Delta^*S^1(u) + q_2\Delta^*S^1(d) = 0 \]

where \( q_1 + q_2 = 1 \), and \( q_1 > 0, q_2 > 0 \).

In shorthand, we write this:

\[ Q \cdot \Delta^*S^1 = 0, \]

where \( S^1 \) gives the first column of the gain/loss matrix, and so on.

**Example.** Find the risk-neutral probability vector, if it exists, for

\[ M = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix} \]

\[ MX = \begin{bmatrix} V_u \\ V_d \end{bmatrix}, V_0 = 0. \]

**Discussion.** Let us first check to see if the market is a “good one.” Recall that

\[ W(\subseteq \mathbb{R}^2) = \{ V^*_1 \in \mathbb{R}^2 \mid xS^1_0 + y^*1 = 0 \}. \]

Thus, we have the equation for \( W \):

\[ 5x + y = 0, \]

or,

\[ y = -5x \text{ (a line)}. \]
We can see that the market is arbitrage-free with this $S_0$.

To find the risk-neutral probability vector, we set up

$$Q^T M^* = S_0^T$$

$$Q^T \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix} = (5 \ 1)$$

$q_1 > 0, q_2 > 0 \begin{cases} 6q_1 + 4q_2 = 5 \\ q_1 + q_2 = 1 \end{cases}$

$\Rightarrow q_1 = q_2 = \frac{1}{2}$.

[Worksheet 3]
Lecture 7. The Big Results for Single-Period Markets, and the Arbitrage Pricing Theorem

Quick review of single period markets:

- The forward problem:
  \[ M X = V_1 \]  
  (m states, n assets, M is (m×n))
  \[ \uparrow \]
  solution is called a hedge or replicating portfolio for \( V_1 \)

- Using the discounted payoff form:
  \[ M^* = \frac{1}{1+r} M \]
  \[ V_{1*} = \frac{1}{1+r} V_1 \]

  The forward problem then has the form:
  \[ M^* X = V_{1*} \]

- We started talking a few weeks ago about Pricing Derivatives. If a call option \( V_1 \) (call) is attainable (i.e., it can be hedged) in a given market, then we can determine how to price it by following this procedure:
  i. Set up \( M X = V_1 \) (call)
  ii. The price of the call is \( V_0 = X^T S_0 \) (or, in other notation, \( V_0 = X \cdot S_0 \)) with the last entry of \( S_0 \) normalized for bonds.
  From this, the strike price is K = V_0 and
  \[ V_1 \) (call) = \[ \begin{bmatrix} [S_{1*}^u - K]_+ \\ [S_{1*}^d - K]_+ \end{bmatrix} \]

  There is an analogous form for puts.

- Our primary objective for this lecture is to work out the Arbitrage Pricing Formula:
  o If we have a call \( V_1 \) (call), then
  \[ V_0 = E_Q[V_1^*] \]
  \[ = Q^T V_{1*} = \sum_{j=1}^{m} q_j \frac{V_1(\omega_j)}{(1+r)} = \frac{1}{(1+r)} E_Q[V_1]. \]
  \[ \uparrow \]
  (undiscounted)
Here, \( Q \) does not have to be risk-neutral. Therefore we cannot take for granted that \( V_0 \equiv X^T S_0 \). The result is that \( M \) is arbitrage-free if and only if \( M \) has at least one risk-neutral probability vector \( Q \). \( Q \) solves this equation:

\[
Q^T M = (1 + r) S_0^T
\]

or, by diving by \((1 + r)\),

\[
Q^T M^* = S_0^T.
\]

Suppose

\[
M = \begin{bmatrix}
6 & 10 \\
4 & \frac{10}{9} \\
3 & \frac{10}{9} \\
\end{bmatrix}
\]

What is the payoff form?

\[
\rightarrow M^* = \begin{bmatrix}
\frac{9+6}{10} & 1 \\
\frac{9+4}{10} & 1 \\
\frac{9+3}{10} & 1 \\
\end{bmatrix}
\]

payoff form

Finding an Arbitrage

- We can show that a system has arbitrage by showing that there is no \( Q \) - the risk-neutral probability vector to the solution.
- Example:

\[
M^* = \begin{bmatrix}
6 & 12 & 1 \\
6 & 8 & 1 \\
4 & 8 & 1 \\
\end{bmatrix}, \quad Q^T M^* = [5, 10, 1]
\]

Hint: Try \( q_2 = 0 \).

\[
\begin{cases}
6q_1 + 6q_2 + 4q_3 = 5 \\
12q_1 + 8q_2 + 8q_3 = 10 \\
q_1 + q_2 + q_3 = 1
\end{cases}
\]

Check the determinant. \( \text{Det}(M^*) = -8 \neq 0 \), so any solution for \( Q \) we find must be unique. Thus, if there is a \( Q \), it must have \( q_2 = 0 \). This violates the condition that \( q_j > 0 \) for a risk-neutral probability vector.
Since there is no risk-neutral probability vector, there is arbitrage. If you want a free lunch, go find the arbitrage (assignment for next time).

**Definition of Arbitrage** for \((M, S_0)\)- a hedge \(X\) such that \(MX = V_1\) where three properties are satisfied:

i. \(V_0 = X^T S_0 = 0\)
ii. \(V_1 \geq 0\) (payoff vector non-negative)
iii. \(V_1 \neq 0\) (one component strictly positive)

Hint to find arbitrage: use the definition above and think about arbitrage in terms of this geometry:

- We will use the facts that given a market, no arbitrage implies \(W \cap A = \emptyset\), and \(W \perp \cap P \perp \neq \emptyset\), and vice versa. From the \(M^*\) form we must learn how to calculate \(W\) via solutions to \(MX = V_1\), and then from \(W\) construct \(W \perp\). Once we have \(W \perp\), it should not be difficult to find an example of \(A\).
  But first, let us answer some leading questions.

- **Question 1.** It is given that \((M, S_0)\) is arbitrage-free, but it’s not very nice, because it has two \(Q\)’s:

  \[ Q \neq Q^1. \]

  Where is the hitch, financially?
Discussion. By the Arbitrage Pricing Formula,

\[ V_0 = E_Q[V_1^*] \] (for a given discounted claim \( V_1^* \))

and

\[ V_0 = E_Q^1[V_1^*]. \]

Thus, \( E_Q[V_1^*] = E_Q^1[V_1^*] \) are always equal for attainable payoffs.

- **Question 2.** Given \( \overline{M}^* \) and \( S_0 \)

\[
\overline{M}^* = \begin{bmatrix} 6 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix}, \quad S_0^T = (5, 1).
\]

why is it arbitrage-free?

Discussion. We can see that the market is incomplete, since \( \overline{M} \) is a (3×2) matrix. However, a matrix can have no arbitrage yet still be incomplete.

Looking at \( Q^T \) in \( Q^T \overline{M}^* = S_0^T \).

\[
(q_1, q_2, q_3) \begin{bmatrix} 6 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix} = (5, 1)
\]

yields

\[
\{Q\} = \{ (\lambda, 2 - 3\lambda, -1 + 2\lambda) | \ 1/2 < \lambda < 2/3 \}.
\]

*Only now* we know that \((\overline{M}, S_0)\) is incomplete. Having found at least one \( Q \) it is guaranteed to be arbitrage-free.

- **Question 3.** Using the same \((\overline{M}, S_0)\), can we find a particular example or set of examples of a payoff \( V_1^* \) that is not attainable (that cannot be hedged)?

In other words, show that there is no \( X \) such that \( \overline{M}^* X = V_1^* \).

Discussion.

\[
\overline{M}^* X = V_1^*
\]

\[
\begin{bmatrix} 6 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_1^u \\ V_1^m \\ V_1^d \end{bmatrix} \quad (3 \text{ states, } 2 \text{ assets})
\]

The system of equations:
{
\begin{align*}
6x_1 + x_2 &= V_1^u \\
4x_1 + x_2 &= V_1^m \\
3x_1 + x_2 &= V_1^d
\end{align*}

Matt’s equation:

\[ \Rightarrow \begin{cases} 
  x_2 = V_1^u - 6x_1 \\
  x_2 = V_1^m - 4x_1 \\
  x_2 = V_1^d - 3x_1
\end{cases} \]

Thus, only this set is hedgeable:

\[
\left( \frac{V_1^u - V_1^m}{2} \right) = \left( \frac{V_1^u - V_1^d}{3} \right) = V_1^m - V_1^d.
\]

(This can be re-written in the form \( AV_1^u + BV_1^m + CV_1^d = D \), where \( D \) can be zero.)

The system is incomplete, but there is no arbitrage. \( W \perp \) will also be a function of \( A \), \( B \), and \( C \), and is orthogonal to the plane \( AV_1^u + BV_1^m + CV_1^d = 0 \).

We will see later that \( P \perp \) is another hyperplane and \( W \) is, in fact, one of the “walls” in this example.

**Important:** Remember the three Big Results:
• **Big Result 1.** \((\mathbf{M}, S_0)\) is arbitrage-free if and only if there exists at least one risk-neutral probability vector \(Q\).

• **Big Result 2: The Arbitrage Pricing Formula.** If \((\mathbf{M}, S_0)\) is arbitrage-free and \(V_1\) is attainable, then this also holds:

\[
V_0 = E_Q[V_1^*]
\]

  o **A quick proof:**

  Given:

  i. Since \((\mathbf{M}, S_0)\) is arbitrage free, then there is at least one \(Q\) such that \(Q^\top \mathbf{M}^* = S_0^\top\).

  ii. Since \(V_1\) is attainable in this market, then there is an \(X\) such that \(\mathbf{M}^* X = V_1^*\).

Thus,

\[
V_0 = X^\top S_0 = S_0^\top X = Q^\top \mathbf{M}^* X = Q^\top V_1^* = E_Q[V_1^*].
\]

• **Big Result 3.** If \(V_1\) is attainable in \((\mathbf{M}, S_0)\), and \(Q \neq Q^1\) are two risk-neutral probability vectors, then

\[
V_0 = E_Q[V_1^*] = E_{Q^1}[V_1^*].
\]

  o **Another quick proof:**

  Given:

  i. Since \(V_1\) is attainable in the given market, then there exists some \(X\) such that \(\mathbf{M}^* X = V_1^*\).

  ii. Since there exists some \(Q \neq Q^1\), then \((\mathbf{M}, S_0)\) is arbitrage-free and

\[
Q^\top \mathbf{M}^* = S_0^\top = Q^\top Q^1 \mathbf{M}^*.
\]

Thus, we have that

\[
V_0 = X^\top S_0 = E_Q[V_1^*]
\]

\[
= X^\top S_0 = E_{Q^1}[V_1^*].
\]

So, we have the same payoff despite different risk-neutral methods of pricing.

• **Example.** Trying to price calls in an incomplete market:

  o Using an incomplete 3×2 market, and given a \(V_1\) and strike price:

\[
\mathbf{M}^* = \begin{bmatrix}
6 & 1 \\
4 & 1 \\
3 & 1
\end{bmatrix}, \quad K = 5, \quad V_1 = \begin{bmatrix}
[6 - 5]^+ \\
[4 - 5]^+ \\
[3 - 5]^+
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

  Method 1 of pricing: Solve for the hedge.
\[
\begin{bmatrix}
6 & 1 \\
4 & 1 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{aligned}
6x_1 + x_2 &= 1 \\
4x_1 + x_2 &= 0 \quad \Rightarrow x_1 = x_2 = 0; \text{ not hedgeable}
\end{aligned}
\]

- Now try \( K = 3.5 \):

\[
V_1 = \begin{bmatrix}
2.5 \\
.5 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
6 & 1 \\
4 & 1 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
2.5 \\
.5 \\
0
\end{bmatrix}
\]

\[
\begin{aligned}
6x_1 + x_2 &= 2.5 \\
4x_1 + x_2 &= .5 \quad \Rightarrow \text{ Also not hedgeable. Only trivial calls can be hedged.}
\end{aligned}
\]

- A trivial call:

\[
\begin{bmatrix}
6 & 1 \\
4 & 1 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

\( x_1 = 0, x_2 = 1 \) solves it for this non-random payoff (it’s a bond!)