Pricing multiple hedges

When \((M, S_0)\) is incomplete

say \(\text{rank}(M) < m < n\) and \(\forall \in \mathbb{R}^m\)

is hedgeable, then there are \(\infty\)

many hedges \(\bar{X}'(\bar{V}) = \bar{X}(\bar{V}) + \sum \beta_j X_j^{(0)}\)

where \(\{X_j^{(0)}\}_{j=1}^k\) is a basis for \(\text{Null}(M)\).

pf: \(\bar{M} \bar{X}' = \bar{M} \bar{X} + \sum \beta_j \bar{M} X_j^{(0)} = \bar{V} + \bar{\alpha}\).

Prop The n.a.s.c. for a unique replication

price for hedgeable \(\forall\) in the above

setting is \(\sqrt{N(M)} \perp \bar{V}\)

pf: \(V_0(\bar{X}'(\bar{V})) = \int_0^T \bar{X}' = \int_0^T \bar{X} + \sum \beta_j \int_0^T S_0 \bar{t} X_j^{(0)}\)

\(= V_0(\bar{X}(\bar{V})) \iff \int_0^T S_0 \bar{t} X_j^{(0)} = 0 \quad \forall j = 1, ..., k\).

When is n.a.s.c. in above

prop. valid?

Assume as before \(\text{rank}(M) < m < n\)

The market is in complete

and there are multiple hedges
hedges \( X'(r) \) for hedges of

\[ \mathbf{v} \in \text{colspan } (M) . \]

By well known th.

\[ N(M) \oplus \text{Re}(M^t) = \mathbb{R}^n \]

Thus, since \( \dim N(M) = \text{nullity } v \equiv M = k \geq 1 \) we must have

\[ \dim \text{Re}(M^t) = n - k < n . \]

where \( \text{Re}(M^t) \) is the

Range \((M)\) which is a subspace of \( \mathbb{R}^n \).

By the dual \((***)\) problem

for \( R-N \) measure \( \overline{Q} \in \mathbb{R}^m \) there

\[ \sum q_j = 1 \quad (***) \quad M^t \overline{Q} = (1+r \overline{r}) \overline{r} \]
There is a vector $\vec{x} \in \mathbb{R}^{m}$
for given initial price vector iff $\vec{s}_{0} \in \mathbb{R}_{+}(m^{t})$. Thus, the existence of a RN $\vec{x} \geq 0$
$s_{0}^{t} \to (x^{t}) \Rightarrow \vec{s}_{0} \in \mathbb{R}_{+}(m^{t})$.

Since $\mathbb{R}^{n}$ is an inner product vector space (armed with the usual dot/scalar product), $\mathbb{R}_{+}(m^{t}) \perp N(m^{\perp})$. Thus the existence of a RN $\vec{x} \geq 0$
implies $\vec{s}_{0} \perp N(m^{\perp})$ and we have finished the proof of
There is a vector $\overrightarrow{a} \in \mathbb{R}^n$ for given initial price vector $\overrightarrow{v}$ iff $\overrightarrow{s}_0 \in \text{Re}(\overrightarrow{m}^T)$. Thus, the existence of a RN $\overrightarrow{a} \geq 0$

$s_0^n \rightarrow (\forall x) \Rightarrow \overrightarrow{s}_0 \in \text{Re}(\overrightarrow{m}^T)$

Since $\mathbb{R}^n$ is an inner product vector space (armed with the usual dot/scalar product), $\text{Re}(\overrightarrow{m}^T) \perp \text{N}(\overrightarrow{m})$. Thus existence of RN $\overrightarrow{a} \geq 0$ implies $\overrightarrow{s}_0 \in \text{N}(\overrightarrow{m})$ and we have finished the proof of
Then, when \( \text{rank}(M) < m < n \) 
the existence of \( \alpha > 0 \) is sufficient condition for multiple hedges to have a unique replication price
\( V_0(\tilde{X}(\hat{v})) \) for any hedgeable \( \tilde{V} \in \mathbb{R}^m \).

Note that \( \exists \alpha > 0 \) is not a necessary condition for hedges to have a unique price.

For \( s_t \geq 1 \) \( \mathbb{W}(M) \Rightarrow \exists \tilde{Q} \in \mathbb{R}^m \) s.t. \( \sum q_i = 1 \), in (\#x)
Then, when \( \text{rank}(M) < m < n \)
the existence of \( R^N \alpha \gtrsim 0 \)
is sufficient condition for
multiple hedges to have a
unique replication price
\( V_0(\tilde{x}(\tilde{V})) \) for any hedgeable
\( \tilde{V} \in \mathbb{R}^m \).

Note that \( \exists \tilde{\beta} > 0 \) is not a
necessary condition for
hedge to have a unique price.

For \( \tilde{F}_0 \)
\( \mathbb{N}(M) \Rightarrow \exists \tilde{Q} \in \mathbb{R}^m \) s.t. \( \sum \tilde{q}_i = 1 \), in (x.x)
P25. Give an example of a 3x5 market \((M_{3x5}, S_0)\) with rank = 2.

P26. Calculate \(\text{Null}(M_{3x5})\) of the matrix in P25.

P27. Calculate \(\text{Range}(M_{3x5})\) of matrix in P25.

P28. Give an example of a 2x2 market \((M_{2x2}, S_0)\) which is not ARF (arbitrage-free).

P29. Construct an explicit arbitrage in the example in P28.

P30. Give an example of a 2x2 market \((M_{2x2}, S_0)\) which is AF.

P31. Construct an explicit (numerical example) of a 3x3 market \((M_{3x3}, S_0)\) which is complete but not AF.

P32. What is one explicit arbitrage in P31?

P33. Construct a \((M_{3x3}, S_0)\) which is AF.
Consider a 1-period complete market \((M_{\text{mxn}}, \bar{S}_0)\), i.e. \(\text{rank}(M)=m\).

Suppose that it has a RN \(\bar{\alpha} > 0\) solution to
\[
Mt_{\text{mxn}} \bar{\alpha} = (1+r)\bar{S}_0.
\]

From this follows several properties:

(i) \(\bar{S}_0 \in \text{Range}(M^t_{\text{mxn}}) \Rightarrow \bar{S}_0 \perp \text{Null}(M_{\text{mxn}}) \Rightarrow \forall \bar{V} \in \mathbb{R}^m, \text{there is a unique replication price } V_0(\bar{V}) = \bar{S}_0^t \bar{\alpha}(\bar{V})\)

(ii) For any \(\bar{V} \in \mathbb{R}^m\), the replication price agrees with the risk-neutral one, i.e.
\[
V_0(\bar{V}) = \bar{S}_0^t \bar{\alpha}(\bar{V}) = V_0(\bar{V}) = \frac{1}{1+r} E_p[\bar{V}]
\]

(iii) There are no arbitrages in this market.

Proof (i) done above.

(ii) \(\bar{S}_0^t \bar{\alpha}(\bar{V}) = \bar{S}_0^t M_{\text{mxn}} \bar{\alpha} = \frac{\bar{S}_0^t \bar{\alpha}}{1+r} \bar{V} = \frac{1}{1+r} E_p[\bar{V}].
\]

(iii) Let \(\bar{X} \in \mathbb{R}^m\) be such that \(\bar{S}_0^t \bar{\alpha} = 0 \Rightarrow \bar{X} \in \text{Null}(M_{\text{mxn}}) \Rightarrow \bar{V}(\bar{X}) = 0\)
and thus not an arbitrage.

The Converse Thm

Let \((M_{\text{mxn}}, \bar{S}_0)\) be complete and AF, then

\[\text{s.t. } (\star) Mt_{\text{mxn}} \bar{\alpha} = (1+r)\bar{S}_0, \text{there is a Risk-Neutral measure }\]

Sketch of Proof: In the special case \(m=n\), and \(\text{rank}(M)=m=n\), \(\det M \neq 0\), \(N(M) = \mathbb{R}^n\)
so that there are no zero e.vals. Then \(\text{Range}(M^t) = \mathbb{R}^m \Rightarrow \text{for given } \bar{S}_0 \in \mathbb{R}^m, \text{there is a vector } \bar{\alpha} \in \mathbb{R}^m\)
s.t. \(\bar{S}_0^t \bar{\alpha} = (1+r)\bar{S}_0\), but it remains to prove that \(\bar{\alpha} > 0\) and thus a RN prob measure.

Let \(\bar{X} \in \mathbb{R}^n\) s.t. \(\bar{S}_0^t \bar{X} = 0 \Rightarrow \bar{V}(\bar{X}) = M_{\text{nxm}} \bar{X} \in K^c\) \(\text{where } K = \{(\bar{X}) \in \mathbb{R}^n \mid \bar{S}_0^t \bar{X} = 0\}\).

Since \(\bar{X} \neq \bar{0}\), \(\bar{V}(\bar{X}) \in \text{complement of } K\), all such \(\bar{V}(\bar{X}) < \mathbb{R}^n \setminus \{\bar{0}\}\) Since \(\bar{X} \perp \bar{S}_0\) \Rightarrow all such \(\bar{X} \in \mathbb{R}^n\) & \(\text{codim } K = 1\).
 Codim(a) = \frac{1}{\text{dim}(a)}

Then there is a vector $v \in \text{ker}

(\text{Range}(M)} = \mathbb{R}^3$.

Prove that $\text{Range}(M) = \mathbb{R}^3

\Rightarrow$ There is a vector $v \in \text{ker}$ such that $\text{Range}(M) = \mathbb{R}^3$. Since $\text{dim}(\text{Range}(M)) = 3$, $v \neq 0$.

\text{Range}(M)} = \mathbb{R}^3

and normalize $v$. Thus $\|v\| = 1$.

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