Notes Set 2

Contains probs P17 - P24

for Hula 4
Lectures on Martingales, Risk-neutral $\hat{\mathbb{P}}$. 

In a 1-period market $(\mathbb{M}^n, \mathbb{S}_0, \hat{\mathbb{P}})$, a r.v. $X_i$ at $t=1$ (e.g. $\mathbb{S}_1, \mathbb{S}_2, V_1$, etc.) is called a $\hat{\mathbb{P}}$-Martingale if $\hat{\mathbb{E}}_{\hat{\mathbb{P}}}[X_t] = \sum_{j=1}^n p_j X_j(t) = X_0$, its value at $t=1$.

Example 1: Let $S_0 = 1$, $\mathbb{S}_1 = \left( \begin{array}{c} 2 \\ \frac{1}{2} \end{array} \right)$, $\hat{\mathbb{P}} = \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{3} \end{array} \right) \Rightarrow \mathbb{S}_1$ is a $\hat{\mathbb{P}}$-Martingale.

Example 2: Let $S_0 = \$230$, $\mathbb{S}_1 = \left[ \begin{array}{c} 260 \\ 180 \end{array} \right]$, $\hat{\mathbb{P}} = \left[ \begin{array}{c} \frac{2}{3} \\ \frac{1}{3} \end{array} \right] \Rightarrow \mathbb{S}_1$ not a $\hat{\mathbb{P}}$-Martingale.

Problem: Calculate $\hat{\mathbb{P}} = (p_1, p_2)$ such that $p_j > 0$, and $p_1 + p_2 = 1$ for which $\mathbb{S}_1 = \left[ \begin{array}{c} 260 \\ 180 \end{array} \right]$, $S_0 = 230$ is a $\hat{\mathbb{P}}$-Martingale.
The Dual Problem for $(M_{m,n}, \overrightarrow{s}_0)$

\[(\times) \quad \overrightarrow{q^*} M_{m,n} = (1+r) \overrightarrow{s}_0^t \quad \overrightarrow{q^*} = (q_{i1}, \ldots, q_{im})\]

Equivalent to:

\[(\times') M_{m,n}^t \overrightarrow{q^*} = (1+r) \overrightarrow{s}_0^t\]

which is in turn a simultaneous eqns in $m$ unknowns $q_i$.

They are:

1. $E_{\overrightarrow{q^*}}[\overrightarrow{s}_1] = (1+r) \overrightarrow{s}_0$ \(\text{Identify rand. Var}\)
2. $E_{\overrightarrow{q^*}}[\overrightarrow{s}_k] = (1+r) \overrightarrow{s}_0$ \(\text{with price vector}\)

\[\frac{\overrightarrow{q^*_i}}{\overrightarrow{q^*_i}} = 1\]

Solvability of $(\times')$ for a complete 1-period market:

Since $\text{col rank}(M_{m,n}) = \text{row rank}(M_{m,n}) = \text{col rank}(M_{m,n}^t)$

$= m$ by Completeness,

if $n \leq m$, $(\times')$ is solvable for $\overrightarrow{q^*}$.

Def. A Risk-Neutral Prob. measure is a $\overrightarrow{q} \in \mathbb{R}^m$ that satisfies $(\times')$ and $q_j > 0$ for $j=1,\ldots,m$.

Geometrically, one requires $\overrightarrow{q^*}$ to be in the interior (1st Octant) in $\mathbb{R}^m$. 
For a 2x2 complete market
\[ M_{2x2} = \begin{bmatrix} S_1^u & S_1^d \\ S_2^u & S_2^d \end{bmatrix}, \quad S_0^I = (S_0) \]

State \( q \) (x, \( x \)) yields:
\[ q_1 = \frac{(1+r)S_0 - S_1^u}{S_1^u - S_0^d} \]
\[ q_2 = \frac{S_1^u - S_0(1+r)}{S_1^u - S_1^d} \]

Note that in order for \( q = (q_1, q_2) \) to be in int(1st quad), n.a.s.c. are
(i) \( S_1^u < (1+r)S_0 \)
(ii) \( (1+r) < S_1^u \)

These algebraic/financial conditions will reappear in the Theory for No-Arbitrage in 1-period markets.

18. Calculate a Risk-Neutral Probability
\[ q = (q_1, q_2) \] with \( q_1 > 0 \) and \( q_2 > 0 \) and \( q_1 + q_2 = 1 \)
for the market AAPL

19. Calculate \( V_0 = E^Q[V] \) where
\[ V \] is the call on AAPL with strike \( K = S_0 = 230 \).
Next we consider the Risk-Neutral Problem geometrically, first for the complete 2×2 market \[ \begin{bmatrix} S_0^u & 1+r \end{bmatrix} \].

\[ M^t \] acts on \( \mathbf{Q} \in \text{int}(1\text{st quad}) \) and sends it to \( \mathbf{Q}_0 = (S_0 > 0) \) also in \( \text{int}(1\text{st quad}) \).

Note that for a complete market, \( \det(M_{2×2}) = \det(M^t_{2×2}) = (S_1^u - S_1^d)(1+r) > 0 \).

In other words, \( M_{2×2} \) and \( M^t_{2×2} \) are orientation-preserving transformations.

**Eigenvalues**

For the 2×2 market that is complete, the char. polyn is

\[ 0 = \det(M - \lambda I) = \det \begin{bmatrix} S_0^u - \lambda & 1+r \\ S_0^d & 1+r - \lambda \end{bmatrix} \]

\[ = (S_0^u - \lambda)(1+r - \lambda) - S_0^d(1+r) \]

\[ = \lambda^2 - \lambda(S_0^u + 1+r) + (S_0^u - S_0^d)(1+r) \]

\[ = \lambda^2 - \lambda(\text{tr} M_{2×2}) + \det M_{2×2} \]

\[ \lambda = \frac{\text{tr} M_{2×2} \pm \sqrt{\text{tr}^2(M_{2×2}) - 4D}}{2} \]

where \( D = \det M \).

Thus, eigenvalues are real if \( \text{Discriminant} \geq 4 \det M_{2×2} \).

Under these conditions, there is a similarity transformation

\[ M_{2×2} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \]

General Question: What is the financial role of any of 
\( \lambda_1, \lambda_2 \in \mathbb{R} \).
Besides the hedging prob. \((*)\) \(M \overset{\rightarrow}{x}=\hat{\sigma}^\top\) used to calculate a hedging portfolio \(\hat{x}(\hat{\sigma})\) for payoff \(\hat{\sigma} \in \mathbb{R}^m\), there is another significant linear system equation \((**\ast)\) that is important in finance: \(\hat{\sigma}^\top M = (1+r)\overset{\rightarrow}{S}_0^\top \) \((**\ast)\)

where \(\hat{\sigma}^\top = (q_1, \ldots, q_m) \in \mathbb{R}^m\) is a left-soln to the matrix of the market.

More constraints on \(\hat{\sigma}^\top\) will be needed:

(I) to make \(\hat{\sigma}^\top\) a probability vector, \(q_j \geq 0\) and \(\sum q_j = 1\)

(II) to make \(\hat{\sigma}^\top\) a risk-neutral probability vector, one needs \(q_j > 0\) and \(\sum q_j = 1\)

Clearly \((**\ast)\) is equivalent to \(M^\top \hat{\sigma}^\top = (1+r)\overset{\rightarrow}{S}_0^\top \) \((**\ast)\)'

It is therefore a system of \(n\) eqns. for \(m\) variables \(q_j\).

Eq 1: Consider market \((M_{2 \times 2} \hat{\sigma}^\top \overset{\rightarrow}{S}_0 \hat{p})\) where \(M_{2 \times 2} = \begin{bmatrix} S_{11} & 1+r \\ S_{21} & 1+r \end{bmatrix}\)

Then \(M^\top \hat{\sigma}^\top = (1+r)\overset{\rightarrow}{S}_0\) \((**\ast)\)'

becomes the simultaneous eqns. \(\begin{cases} q_1 S_{11} + q_2 S_{21} = (1+r)S_0 \\ q_1 + q_2 = 1 \end{cases}\)

Note \((1) \iff q_1 + q_2 = 1\) \((2)\)
\( \text{CN} \Rightarrow \text{gel} \quad q \),

\[ q, S_i^u + (1 - q) S_i^d = (1 + r) S_0 \quad (1) \quad \Rightarrow \quad q_1 = \frac{(1 + r) S_0 - S_i^d}{S_i^u - S_i^d} q_2 = \frac{S_i^u - (1 + r) S_0}{S_i^u - S_i^d} \]

Assuming \( S_i^u > S_i^d \) for completeness.

We note that necessary and sufficient conditions are the same as previously found in the 2x2 market for financial sense of the replication prices of calls, and put on \( S_i \) with strike \( K = S_0 \), namely:

\( \text{a) } S_i^u > (1 + r) S_0 \quad \text{b) } (1 + r) S_0 > S_i^d \).

In general \( 1 \text{-period market } \) \((M = m \times n, \overrightarrow{S_0}, \overrightarrow{P}) \)
we have (\#*)
\[ \overrightarrow{M}_{\text{min}} = (1 + r) \overrightarrow{S_0} \quad \text{or} \quad \overrightarrow{M}^t = (1 + r) \overrightarrow{S_0} \]

which is equivalent to

system of simultaneous \( \varphi \)s:

\( i \quad \sum_{j=1}^{n} q_{ij} \overrightarrow{M}_{j} = (1 + r) \overrightarrow{S_0}^{(n)} = \overrightarrow{q} \cdot \overrightarrow{S}_i = \overrightarrow{q} \cdot \overrightarrow{M}_i \)
\( \cdots \quad \overrightarrow{q} \cdot \overrightarrow{S}_k = \overrightarrow{q} \cdot \overrightarrow{M}_k \)
\( k \quad \sum_{j=1}^{n} q_{jk} \overrightarrow{M}_{j} = (1 + r) \overrightarrow{S_0}^{(k)} \) where \( \overrightarrow{S_0}^{(k)} \) is the \( k \)-th term in \( \overrightarrow{S_0} \).

\( \sum_{i=1}^{n} \overrightarrow{M}_i = 1 \)

\( \text{Here } \overrightarrow{M}_k \equiv k \text{-th col. of } \overrightarrow{M}_{\text{min}} = \overrightarrow{S_k} \text{ price vector at } t = 1 \)

\( S_0 \overrightarrow{M}_n = (1 + r, \ldots, 1 + r) \) is \( t = 1 \) price of bond.
Note that the scalar (inner or dot) product $\mathbf{a} \cdot \mathbf{s}_k$

between the m-vectors $\mathbf{a}$ and $t=1$ price vector $\mathbf{s}_k$ of both asset

$$\mathbf{a} \cdot \mathbf{s}_k = \sum_{j=1}^{m} a_j s_k(j) = E_{\mathbf{a}}[\mathbf{s}_k]$$

when by abuse of notation the price vector $\mathbf{s}_k$ is identified

with the random variable $S_k$, the price at $t=1$ of asset $k$.

Thus, we have shown that $(\star)$ is equivalent to:

$$\frac{1}{1+r} E_{\mathbf{a}}[S_k] = E_{\mathbf{a}}[\mathbf{s}_k] = S_k(0), \text{ for } t=0 \text{, pure } \mathbb{Q} \text{ in asset } k \text{, where } S_k = S_k(t_0)$$

In other words, a solution to $(\star)$ for $\mathbf{a} \in \mathbb{R}^m$

is a probability vector that can be a possible candidate for the (already defined) Martingale/Risk Neutral Probability measure, that simultaneously satisfy (n-1) condition for the Martingale Property of all (n-1) risky asset.
\[ E_y \begin{bmatrix} t \end{bmatrix} \begin{bmatrix} \bar{x}_y \end{bmatrix} = \begin{bmatrix} S_i^y & S_i^y \end{bmatrix} \begin{bmatrix} q_1 \\ 1 \end{bmatrix} = (1+t) \begin{bmatrix} S_0 \\ 1 \end{bmatrix} \] (**)'}

Consider solvability of (**) with fixed vector \( \begin{bmatrix} S_0 \\ 1 \end{bmatrix} = \bar{x}_0 \). Since col. rank \( (M_{m\times n}) = \text{row. rank} \) \( (M_{m\times n}) \)
we deduce that \( M_{2\times 2} \) has 2 linearly independent columns as long as \( S_i^y > S_i^y \). Thus there is always a solution

\[ \bar{x} \in \mathbb{R}^2 \] such that \( q_1 + q_2 = 1 \), \( E_x \begin{bmatrix} S_i \end{bmatrix} = (1+t) S_0 \).

for all \( t=0 \) prior vector \( \bar{x}_0 = \begin{bmatrix} S_0 \\ 1 \end{bmatrix} \).

Note the important point that any solution to (**) for \( \bar{x} \in \mathbb{R}^m \)
does not mean \( q_j > 0 \) for \( j=1, ..., m \). In fact, not even \( q_j > 0 \) for all

Only vectors with \( \bar{x} > 0 \), i.e. \( q_j > 0 \) for all \( j=1, ..., m \) are Risk-Neutral measures. This is captured geometrically by the requirement

that \( \bar{x} \) is in the interior of the 1st Octant (it does not belong to the boundary faces nor axes of 1st Octant).
Given that \((M_{2w,1}, S_0)\) has a Risk Neutral \(\mathbb{Q}\), there are several consequences which follow:

(1) **Thm**: Every \(\bar{V} \in \mathbb{R}^2\) after discounting by \(Y_{1+t}\)

is a \(\mathbb{Q}\)-Martingale.

**Pf**: Existence of RN \(\mathbb{Q}\) \(\Rightarrow \ S_0(1+r) < S_t^u\) and \(S_0 < S_{t^d}\),

which in turn \(\Rightarrow S_t^u > S_t^d\). Thus, by completeness

of \((M_{2w,1}, S_0)\), every \(\bar{V} \in \mathbb{R}^2\) has a hedge \(\tilde{X}(\bar{V})\)

\(S_t + M_{2w,1} \tilde{X} = \bar{V}\) and a replication price

\(SV_0 = S_0 \tilde{X}(\bar{V})\). We now show that

\[E_Q[\bar{V}] = \tilde{X}(\bar{V}) = \tilde{X}(1+r) (1 + r) S_0 \tilde{X}(\bar{V})\]

\[\Rightarrow E_Q[\bar{V}] = \tilde{X}(\bar{V}) = V_0\ \ \ QED\]

(2) **Cor** In \((M_{2w,1}, S_0)\) with RN \(\mathbb{Q}\),

the replication price \(SV_0 = \frac{E_Q[\bar{V}]}{1+r}\)

for each \(\bar{V} \in \mathbb{R}^2\).
Let

\[ M = \begin{bmatrix} 260 & 7.0 \\ 180 & 1.01 \end{bmatrix}, \quad S_0 = \begin{pmatrix} 219 \\ 1 \end{pmatrix} \]

Price can at Strike \( K = 220 \).
**No Arbitrage Principle**

An arbitrage in \((M_{\text{fin}}, \tilde{S}_0, \tilde{P})\) is a portfolio \(\tilde{X}\) st.

(i) \(X_0 = \tilde{S}_0 \tilde{X} = 0\)

(ii) \(\exists \tilde{Q} \geq 0 \quad \tilde{V} = \tilde{m} \tilde{X}\)

(iii) \(X_j > 0\) for at least one \(j = 1, \ldots, m\)

**Example:** In a \(2 \times 2\) case, \((M, \tilde{S}_0)\) complete, an arbitrage is a portfolio

\[
\tilde{X} = (X_1, X_2) \quad \text{s.t.} \quad X_1 \tilde{S}_0 + X_2 = 0 \quad (1)
\]

\[
\tilde{V} > 0 \\
V_1 > 0 \text{ or } V_2 > 0.
\]

**Definition:** Market \((M, \tilde{S}_0)\) is arbitrage-free or AF if there are no arbitrages in it.

**P21:** Calculate the risk price of a call at strike \(K\) st.

\(S_d < K < S_u\) in a \(2 \times 2\) market \((M_{2 \times 2}, \tilde{S}_0)\) for which there is a \(\tilde{Q} > 0\) satisfying \(\tilde{Q} \tilde{m}_{2 \times 2} = (1 + \tilde{r}) \tilde{S}_0\).

**P22:** Do P21 but for a put at same \(K\).
Thm. n.a.s.c. for \( (n_{LX, \rightarrow, S_0}) \)

to be AF is

(i) \( S_0(t + r) < S_i^u \)

(ii) \( S_i^d < S_0(t + r) \)

\[ \text{PF: By defn of an arbitrage } x \in \mathbb{R}^2 \text{ we have} \]

\[ S_0 \cdot x = S_0 x_1 + x_2 = 0 \implies x_2 = -S_0 x_1, \]

Substituting in

\[ \tilde{V}(x) = \nabla_{x_2} \tilde{V}(x) \text{ we get} \]

\[ V_i^u = S_i^u x_1 + (1 + r)(-S_0 x_1) = x_1 (S_i^u - (1 + r) S_0) \geq 0 \quad (1) \]

\[ V_i^d = S_i^d x_1 + (1 + r)(-S_0 x_1) = x_1 (S_i^d - (1 + r) S_0) \leq 0 \quad (2) \]

iff n.a.s.c. (i) + (ii) holds.

Next note that (i) + (ii)

\[ \implies \tilde{x} \notin \text{Oct. 1} = \text{closure of 1st quad.} \]

thus not an arbitrage.
P23: Construct an arbitrary 
\[ \mathbf{x} = (x_1, x_2) \] when

\[ A = S_i^a - S_0(1+r) > 0 \]

\[ B = S_i^a - S_0(1+r) > 0 \]

Give answer for \( x_1, x_2 \) in terms of \( A, B \)

P24: Give conditions in terms of \( A, B \) in P23 in order for \((M_{2x2}, S_0)\) in P23 to be complete