Thm 1: A market \((M, \vec{S})\) is complete iff

\[ \text{rank} \left( M \right) \geq m, \text{ the } \# \text{ of market states} \]

Proof: Suppose \( M \) has rank \( \geq m \), then

\[ \{ M \vec{x} | \vec{x} \in \mathbb{R}^n \} = \text{span}(M) \text{ has dim } \geq m. \]

So each \( \vec{v} \in \mathbb{R}^m \) is in this vector space, the complete, i.e. every claim is replicable.

Suppose \( M \) complete, then \( \text{span}(M) \geq \mathbb{R}^m \), then

\[ \text{rank}(M) = \dim(\text{span}(M)) \geq m \ \square. \]
Defn: An arbitrage in single-period market $(\overrightarrow{M}, \overrightarrow{S_0})$ is a portfolio-clair pair $(\overrightarrow{x}, \overrightarrow{V})$
\[ \text{s.t. } (a) \, M \cdot \overrightarrow{x} = \overrightarrow{V}, \quad (b) \, V_0 = \overrightarrow{x} \cdot \overrightarrow{S_0} = 0, \quad (c) \, V_j \geq 0, \; j=1,\ldots, m \]
(d) $V_k > 0$ for some k

Defn: A market $(\overrightarrow{M}, \overrightarrow{S_0})$ is arbitrage-free or AF if there are no arbitrage in it.

Examples of $2 \times 2$ $M = \begin{bmatrix} S^u & 1+r \\ S^d & 1+r \end{bmatrix}$, $\overrightarrow{S_0} = (S_0)$
\[ \text{s.t. } (\overrightarrow{M}, \overrightarrow{S_0}) \text{ is complete and AF} \]

Prop: Market $(\overrightarrow{M}_{2 \times 2}, \overrightarrow{S_0})$ is complete and AF iff
\[ \text{ (a) } S^u > S_0 (1+r), \quad (b) S^d < S_0 (1+r) \]

Numerical example with arbitrage:
$M = \begin{bmatrix} 170 & 1:1 \\ 150 & 1:1 \end{bmatrix}$, $\overrightarrow{S_0} = (160, 1)^t$
Proof of Prop on p. 4: \( S_0 (1+r) \leq S_i^d < S_i^u \)

Suppose \( S_i^d < S_i^u \leq S_0 (1+r) \Rightarrow (M, f^*) \) is complete.

Consider \( (X) \setminus \mathbb{N} \sim \sqrt{V} \) or equiv.,

1. \( S_i^d x_s + (1+r) x_b = V^u \Rightarrow x_s = \frac{V^u - V^d}{S_i^d - S_i^u} \quad x_b = \frac{V^u - S_i^u x_s}{1+r} \)
2. \( S_i^d x_s + (1+r) x_b = V^d \) is called the delta hedge

Consider \( x = (x_s, x_b)^t \) s.t. \( x^t S_0 = x_s S_0 + x_b = 0 \), i.e.,

\( x_b = -x_s S_0 \) or choosing \( x_s = \sqrt{V} \) then \( x_b = -\sqrt{V} \).

\(-S_i^d + (1+r) S_0 = V^u \geq 0 \) satisfying condition of arbitrage

\(-S_i^d + (1+r) S_0 = V^d \geq 0 \)

Since \( S_i^d < S_i^u \), \( V^d \geq 0 \), so we have our arbitrage.

Suppose \( (M, f^*) \) is complete and AF, then

\( S_i^d < S_i^u \) but \( S_i^d < S_0 (1+r) < S_i^u \).
Proof #2 of Prop. on p. 4

Consider the adjoint problem: for \( \bar{\mathbf{q}} = (q^u, q^d)^t \)

\[
(q^u, q^d) W = (1+r)S_0 \tag{**} \]

which is equiv: (1) \( \mathbb{E}_q [S_t] = S_t^u q^u + S_t^d q^d = (1+r) S_0 \)

(2) \( q^u + q^d = 1 \).

Solution of (**) is called the virtual, or Martingale, prob. meas. if it exists. Thus (1) states that \( S_t \) is a Martingale Random Variable.

Lemma 1: Conditions (a) + (b) for complete and AF are equivalent to existence of \( \bar{\mathbf{q}} \) in (**).

pf: (**) is equivalent to \( W^t \bar{\mathbf{q}} = (1+r) S_0 \) \( \tag{**} \)

(**) has a soln \( \bar{\mathbf{q}} \) iff \( S_0 \in \text{Range}(W) \perp \text{Range}(M) \).

The 2nd cond in (**), \( q^u > 0, q^d > 0 \)

\[ q^u S_t^u + (1-q^u) S_t^d = (1+r) S_0 \implies q^u = \frac{(1+r) S_0 - S_t^d}{S_t^u - S_t^d} > 0 \]

\[ q^d = \frac{S_t^u - (1+r) S_0}{S_t^u - S_t^d} > 0 \]

Numerators of \( q^u > 0, q^d > 0 \)
are resp. (a) + (b).
Proof #3 of Prop. on P. 4: or its equiv. form.

Prop.: \( (M_{2 \times 2}, \mathcal{F}_0) \) is complete + AF iff.

\((\forall x)\), has SOLn \( q^u > 0, q^d > 0 \) (a Martingale m.m.)

PF: Consider the geometry of ARBITRAGE.

\[ \begin{array}{c}
(0,1) \\
-1
\end{array} \]

An Arbitrage is \( (\overline{x}, \overline{V}) \) s.t. \( \overline{x} \cdot \overline{S}_0 = 0 \),

\( \overline{V} \in \{1st and 3rd quadrant\} \setminus \{0\} \). Thus to be AF, a complete \( (M_{2 \times 2}, \mathcal{F}_0) \) has to satisfy cond.: for all \( \overline{x} \in \mathbb{R}^+ \), 
\( \overline{S}_0 \parallel \overline{V} = m \overline{x} \) is in the open set 2nd Quad \( \cup \) 4th Quad

And (ii) Existence of Martingale m.m. \( q^u > 0, q^d > 0 \)

(\( \forall x \)): \( m \overline{x} = (1+r) \overline{S}_0 \).

\( q^d = (q^u > 0, q^d > 0) \) and \( q^u + q^d = 1 \).

Suppose \( \exists \overline{x} > 0 \) (m.m. then any \( \overline{x} \) \parallel \overline{S}_0 \) has \( \overline{V} = m \overline{x} \) \parallel \overline{S}_0 \).

\( \Rightarrow \overline{V} \) in open set 2nd \& 4th quad. \( \Rightarrow \) AF.

Suppose AF, then since \( (M, \mathcal{F}) \) is complete, (\( \mathcal{F}_0 \)) always have \( \mathcal{F}_0 = \sigma(\mathcal{F}) \). 

For all \( \overline{x} \), \( \overline{x} \parallel \overline{S}_0 \) for all \( \overline{x} \cdot \overline{S}_0 = 0 \). Since \( \mathcal{F}_0 \) m.m. \( \overline{V} \) is in \( K \times \cap \) in interior of \( K \).