HWk 4 and 5 on Chapter 2
For Exam 3 (Oct 10)
due to grader: Oct 2
Review/tutorial Oct 5.
Red circles to be graded + exam
Blue circles for discussion + exam

Please NOTE: (1) Office hrs 1-3pm Weds
(Tues hrs cancelled due to conflicts)
(2) NO LMS/rare lect notes posted
By intention to encourage class attendance and participation
(3) There is no difference as far as this course is concerned between the 2nd and later eds of Axler — relevant hwk pages posted
EXERCISES 2.A

1. Suppose \( v_1, v_2, v_3, v_4 \) spans \( V \). Prove that the list
\[
v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4
\]
also spans \( V \).

2. Verify the assertions in Example 2.18.

3. Find a number \( t \) such that
\[
(3, 1, 4), (2, -3, 5), (5, 9, t)
\]
is not linearly independent in \( \mathbb{R}^3 \).

4. Verify the assertion in the second bullet point in Example 2.20.

5. (a) Show that if we think of \( \mathbb{C} \) as a vector space over \( \mathbb{R} \), then the list
\[
(1 + i, 1 - i)
\]
is linearly independent.
   (b) Show that if we think of \( \mathbb{C} \) as a vector space over \( \mathbb{C} \), then the list
\[
(1 + i, 1 - i)
\]
is linearly dependent.

6. Suppose \( v_1, v_2, v_3, v_4 \) is linearly independent in \( V \). Prove that the list
\[
v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4
\]
is also linearly independent.

7. Prove or give a counterexample: If \( v_1, v_2, \ldots, v_m \) is a linearly independent list of vectors in \( V \), then
\[
5v_1 - 4v_2, v_2, v_3, \ldots, v_m
\]
is linearly independent.

8. Prove or give a counterexample: If \( v_1, v_2, \ldots, v_m \) is a linearly independent list of vectors in \( V \) and \( \lambda \in \mathbb{F} \) with \( \lambda \neq 0 \), then \( \lambda v_1, \lambda v_2, \ldots, \lambda v_m \) is linearly independent.

9. Prove or give a counterexample: If \( v_1, \ldots, v_m \) and \( w_1, \ldots, w_m \) are linearly independent lists of vectors in \( V \), then \( v_1 + w_1, \ldots, v_m + w_m \) is linearly independent.

10. Suppose \( v_1, \ldots, v_m \) is linearly independent in \( V \) and \( w \in V \). Prove that if \( v_1 + w, \ldots, v_m + w \) is linearly dependent, then \( w \in \text{span}(v_1, \ldots, v_m) \).
EXERCISES 2.B

1. Find all vector spaces that have exactly one basis.

2. Verify all the assertions in Example 2.28.

3. (a) Let $U$ be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$  

Find a basis of $U$.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace $W$ of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

4. (a) Let $U$ be the subspace of $\mathbb{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$  

Find a basis of $U$.

(b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$.

(c) Find a subspace $W$ of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

5. Prove or disprove: there exists a basis $p_0, p_1, p_2, p_3$ of $\mathcal{P}_3(F)$ such that none of the polynomials $p_0, p_1, p_2, p_3$ has degree 2.

6. Suppose $v_1, v_2, v_3, v_4$ is a basis of $V$. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of $V$.

7. Prove or give a counterexample: If $v_1, v_2, v_3, v_4$ is a basis of $V$ and $U$ is a subspace of $V$ such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then $v_1, v_2$ is a basis of $U$.

8. Suppose $U$ and $W$ are subspaces of $V$ such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of $U$ and $w_1, \ldots, w_n$ is a basis of $W$. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of $V$. 

EXERCISES 2.C

1. Suppose $V$ is finite-dimensional and $U$ is a subspace of $V$ such that $\dim U = \dim V$. Prove that $U = V$.

2. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin.

3. Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

4. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(6) = 0\}$. Find a basis of $U$.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.
   (c) Find a subspace $W$ of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

5. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$. Find a basis of $U$.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.
   (c) Find a subspace $W$ of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

6. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5)\}$. Find a basis of $U$.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.
   (c) Find a subspace $W$ of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

7. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5) = p(6)\}$. Find a basis of $U$.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.
   (c) Find a subspace $W$ of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

8. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$. Find a basis of $U$.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.
   (c) Find a subspace $W$ of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

9. Suppose $v_1, \ldots, v_m$ is linearly independent in $V$ and $w \in V$. Prove that $\dim \text{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$.

10. Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ are such that each $p_j$ has degree $j$. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

11. Suppose that $U$ and $W$ are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$. 
12 Suppose $U$ and $W$ are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

13 Suppose $U$ and $W$ are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

14 Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of $V$. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$ 

15 Suppose $V$ is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of $V$ such that

$$V = U_1 \oplus \cdots \oplus U_n.$$ 

16 Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of $V$ such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$ 

[The exercise above deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a set is written as a disjoint union of finite subsets, then the number of elements in the set equals the sum of the numbers of elements in the disjoint subsets.]

17 You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if $U_1, U_2, U_3$ are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2 + U_3)$$

$$= \dim U_1 + \dim U_2 + \dim U_3$$

$$- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3)$$

$$+ \dim(U_1 \cap U_2 \cap U_3).$$

Prove this or give a counterexample.
HWk 5 Axler Chap 3.
Circled problems in 3A, 3B and extra probs here attached below.

Due Oct 10. This material will be on week of Oct 2 and reviewed in tutorial on Thu Oct 5. It will also be part of exam 3 on Oct 10.

Extra: (Not graded)
Let $U \leq V$, $W \leq V$ be subspaces of $V$. Construct a basis for $U + W \leq V$. 
3.11 Linear maps take 0 to 0
Suppose \( T \) is a linear map from \( V \) to \( W \). Then \( T(0) = 0 \).

**Proof** By additivity, we have

\[
T(0) = T(0 + 0) = T(0) + T(0).
\]

Add the additive inverse of \( T(0) \) to each side of the equation above to conclude that \( T(0) = 0 \).

**EXERCISES 3.A**

1. Suppose \( b, c \in \mathbb{R} \). Define \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) by

\[
T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).
\]

Show that \( T \) is linear if and only if \( b = c = 0 \).

2. Suppose \( b, c \in \mathbb{R} \). Define \( T : \mathcal{P}({\mathbb{R}}) \to \mathbb{R}^2 \) by

\[
Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x) \, dx + c \sin p(0)\right).
\]

Show that \( T \) is linear if and only if \( b = c = 0 \).

3. Suppose \( T \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n) \). Show that there exist scalars \( A_{j,k} \in \mathbb{F} \) for \( j = 1, \ldots, m \) and \( k = 1, \ldots, n \) such that

\[
T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)
\]

for every \( (x_1, \ldots, x_n) \in \mathbb{F}^m \).

*The exercise above shows that \( T \) has the form promised in the last item of Example 3.4.*

4. Suppose \( T \in \mathcal{L}(V, W) \) and \( v_1, \ldots, v_m \) is a list of vectors in \( V \) such that \( T(v_1), \ldots, T(v_m) \) is a linearly independent list in \( W \). Prove that \( v_1, \ldots, v_m \) is linearly independent.

5. Prove the assertion in 3.7.

6. Prove the assertions in 3.9.
Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if \( \dim V = 1 \) and \( T \in \mathcal{L}(V, V) \), then there exists \( \lambda \in F \) such that \( Tv = \lambda v \) for all \( v \in V \).

Give an example of a function \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) such that
\[
\varphi(av) = a\varphi(v)
\]
for all \( a \in \mathbb{R} \) and all \( v \in \mathbb{R}^2 \) but \( \varphi \) is not linear. [The exercise above and the next exercise show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map.]

Give an example of a function \( \varphi : C \to C \) such that
\[
\varphi(w + z) = \varphi(w) + \varphi(z)
\]
for all \( w, z \in C \) but \( \varphi \) is not linear. (Here \( C \) is thought of as a complex vector space.) [There also exists a function \( \varphi : \mathbb{R} \to \mathbb{R} \) such that \( \varphi \) satisfies the additivity condition above but \( \varphi \) is not linear. However, showing the existence of such a function involves considerably more advanced tools.]

Suppose \( U \) is a subspace of \( V \) with \( U \neq V \). Suppose \( S \in \mathcal{L}(U, W) \) and \( S \neq 0 \) (which means that \( Su \neq 0 \) for some \( u \in U \)). Define \( T : V \to W \) by
\[
Tv = \begin{cases} 
Sv & \text{if } v \in U, \\
0 & \text{if } v \in V \text{ and } v \notin U.
\end{cases}
\]
Prove that \( T \) is not a linear map on \( V \).

Suppose \( V \) is finite-dimensional. Prove that every linear map on a subspace of \( V \) can be extended to a linear map on \( V \). In other words, show that if \( U \) is a subspace of \( V \) and \( S \in \mathcal{L}(U, W) \), then there exists \( T \in \mathcal{L}(V, W) \) such that \( Tu = Su \) for all \( u \in U \).

Suppose \( V \) is finite-dimensional with \( \dim V > 0 \), and suppose \( W \) is infinite-dimensional. Prove that \( \mathcal{L}(V, W) \) is infinite-dimensional.

Suppose \( v_1, \ldots, v_m \) is a linearly dependent list of vectors in \( V \). Suppose also that \( W \neq \{0\} \). Prove that there exist \( w_1, \ldots, w_m \in W \) such that no \( T \in \mathcal{L}(V, W) \) satisfies \( T v_k = w_k \) for each \( k = 1, \ldots, m \).

Suppose \( V \) is finite-dimensional with \( \dim V \geq 2 \). Prove that there exist \( S, T \in \mathcal{L}(V, V) \) such that \( ST \neq TS \).
EXERCISES 3.B

1. Give an example of a linear map $T$ such that $\dim \ker T = 3$ and $\dim \text{range } T = 2$.

2. Suppose $V$ is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that
\[
\text{range } S \subset \ker T.
\]
Prove that $(ST)^2 = 0$.

3. Suppose $v_1, \ldots, v_m$ is a list of vectors in $V$. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by
\[
T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.
\]
(a) What property of $T$ corresponds to $v_1, \ldots, v_m$ spanning $V$?
(b) What property of $T$ corresponds to $v_1, \ldots, v_m$ being linearly independent?

4. Show that
\[
\{ T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \ker T > 2 \}
\]
is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

5. Give an example of a linear map $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that
\[
\text{range } T = \ker T.
\]

6. Prove that there does not exist a linear map $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ such that
\[
\text{range } T = \ker T.
\]

7. Suppose $V$ and $W$ are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{ T \in \mathcal{L}(V, W) : T \text{ is not injective} \}$ is not a subspace of $\mathcal{L}(V, W)$.

8. Suppose $V$ and $W$ are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{ T \in \mathcal{L}(V, W) : T \text{ is not surjective} \}$ is not a subspace of $\mathcal{L}(V, W)$.

9. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in $V$. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in $W$. 
Hwk #7  Due to grader Mon Oct 16
For tutorial discussion, Thu Oct 12

Relevant to Exam 4 which will be held 4 weeks from Oct 9. Exam 4 will be based on Hwks 7, 8, 9 all in Chap. 3.
Suppose \( v_1, \ldots, v_n \) spans \( V \) and \( T \in \mathcal{L}(V, W) \). Prove that the list \(Tv_1, \ldots, Tv_n \) spans range \( T \).

Suppose \( S_1, \ldots, S_n \) are injective linear maps such that \( S_1S_2 \cdots S_n \) makes sense. Prove that \( S_1S_2 \cdots S_n \) is injective.

Suppose \( V \) is finite-dimensional and that \( T \in \mathcal{L}(V, W) \). Prove that there exists a subspace \( U \) of \( V \) such that \( U \cap \text{null} \, T = \{0\} \) and range \( T = \{Tu : u \in U\} \).

Suppose \( T \) is a linear map from \( \mathbb{F}^4 \) to \( \mathbb{F}^2 \) such that

\[
\text{null} \, T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.
\]

Prove that \( T \) is surjective.

Suppose \( U \) is a 3-dimensional subspace of \( \mathbb{R}^8 \) and that \( T \) is a linear map from \( \mathbb{R}^8 \) to \( \mathbb{R}^5 \) such that \( \text{null} \, T = U \). Prove that \( T \) is surjective.

Prove that there does not exist a linear map from \( \mathbb{F}^5 \) to \( \mathbb{F}^2 \) whose null space equals

\[
\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.
\]

Suppose there exists a linear map on \( V \) whose null space and range are both finite-dimensional. Prove that \( V \) is finite-dimensional.

Suppose \( V \) and \( W \) are both finite-dimensional. Prove that there exists an injective linear map from \( V \) to \( W \) if and only if \( \dim V \leq \dim W \).

Suppose \( V \) and \( W \) are both finite-dimensional. Prove that there exists a surjective linear map from \( V \) onto \( W \) if and only if \( \dim V \geq \dim W \).

Suppose \( V \) and \( W \) are finite-dimensional and that \( U \) is a subspace of \( V \). Prove that there exists \( T \in \mathcal{L}(V, W) \) such that \( \text{null} \, T = U \) if and only if \( \dim U \geq \dim V - \dim W \).

Suppose \( W \) is finite-dimensional and \( T \in \mathcal{L}(V, W) \). Prove that \( T \) is injective if and only if there exists \( S \in \mathcal{L}(W, V) \) such that \( ST \) is the identity map on \( V \).

Suppose \( V \) is finite-dimensional and \( T \in \mathcal{L}(V, W) \). Prove that \( T \) is surjective if and only if there exists \( S \in \mathcal{L}(W, V) \) such that \( TS \) is the identity map on \( W \).
Suppose $U$ and $V$ are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T.$$ 

Suppose $U$ and $V$ are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}.$$ 

Suppose $W$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{null } T_1 \subset \text{null } T_2$ if and only if there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$.

Suppose $V$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{range } T_1 \subset \text{range } T_2$ if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2S$.

Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that $D$ is surjective. [The notation $D$ is used above to remind you of the differentiation map that sends a polynomial $p$ to $p'$.

Without knowing the formula for the derivative of a polynomial (except that it reduces the degree by 1), you can use the exercise above to show that for every polynomial $q \in \mathcal{P}(\mathbf{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that $p' = q$.]

Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$. [This exercise can be done without linear algebra, but it’s more fun to do it using linear algebra.]

Suppose $T \in \mathcal{L}(V, W)$, and $w_1, \ldots, w_m$ is a basis of range $T$. Prove that there exist $\varphi_1, \ldots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$$

for every $v \in V$.

Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $u \in V$ is not in null $\varphi$. Prove that $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$.

Suppose $\varphi_1$ and $\varphi_2$ are linear maps from $V$ to $\mathbf{F}$ that have the same null space. Show that there exists a constant $c \in \mathbf{F}$ such that $\varphi_1 = c\varphi_2$.

Give an example of two linear maps $T_1$ and $T_2$ from $\mathbf{R}^5$ to $\mathbf{R}^2$ that have the same null space but are such that $T_1$ is not a scalar multiple of $T_2$. 
Two more ways to think about matrix multiplication are given by Exercises 10 and 11.

**EXERCISES 3.C**

1. Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of $V$ and $W$, the matrix of $T$ has at least $\dim \text{range } T$ nonzero entries.

2. Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of $D$ with respect to these bases is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

[Compare the exercise above to Example 3.34. The next exercise generalizes the exercise above.]

3. Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row $j$, column $j$, equal 1 for $1 \leq j \leq \dim \text{range } T$.

4. Suppose $v_1, \ldots, v_m$ is a basis of $V$ and $W$ is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of $W$ such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row, first column.

[In this exercise, unlike Exercise 3, you are given the basis of $V$ instead of being able to choose a basis of $V$.]

5. Suppose $w_1, \ldots, w_n$ is a basis of $W$ and $V$ is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of $V$ such that all the entries in the first row of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row, first column.

[In this exercise, unlike Exercise 3, you are given the basis of $W$ instead of being able to choose a basis of $W$.]
6 Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \text{range } T = 1$ if and only if there exist a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

7 Verify 3.36.

8 Verify 3.38.

9 Prove 3.52.

10 Suppose $A$ is an $m$-by-$n$ matrix and $C$ is an $n$-by-$p$ matrix. Prove that 

$$(AC)_{j, \cdot} = A_{j, \cdot} C$$

for $1 \leq j \leq m$. In other words, show that row $j$ of $AC$ equals (row $j$ of $A$) times $C$.

11 Suppose $a = (a_1 \cdots a_n)$ is a 1-by-$n$ matrix and $C$ is an $n$-by-$p$ matrix. Prove that 

$$aC = a_1C_{1, \cdot} + \cdots + a_nC_{n, \cdot}.$$ 

In other words, show that $aC$ is a linear combination of the rows of $C$, with the scalars that multiply the rows coming from $a$.

12 Give an example with 2-by-2 matrices to show that matrix multiplication is not commutative. In other words, find 2-by-2 matrices $A$ and $C$ such that $AC \neq CA$.

13 Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose $A$, $B$, $C$, $D$, $E$, and $F$ are matrices whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense. Prove that $AB + AC$ and $DF + EF$ both make sense and that $A(B + C) = AB + AC$ and $(D + E)F = DF + EF$.

14 Prove that matrix multiplication is associative. In other words, suppose $A$, $B$, and $C$ are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

15 Suppose $A$ is an $n$-by-$n$ matrix and $1 \leq j, k \leq n$. Show that the entry in row $j$, column $k$, of $A^3$ (which is defined to mean $AAA$) is 

$$\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.$$
1. Let $T : V \to V$ be a linear operator / map on a finite dim vector space $V$. Show that surjectivity of $T$ implies injectivity of $T$.

2. For the same $T$ as in Q1, show that injectivity of $T$ implies surjectivity.

3. Let $\{v_1, ..., v_k\}$ be linearly independent in $V$, and $w$ in $V$. Prove that if $w$ is not in span$\{v_1, ..., v_k\}$, then $\{v_1 + w, ..., v_k + w\}$ is linearly independent in $V$.

4. Prove or disprove : Let $w \in \text{span}\{v_1, ..., v_k\}$, and $a$ is in the scalar field $F$. Then $\{v_1 - aw, ..., v_k - aw\}$ has more than $k - 2$ linearly independent vectors.
Multiplying a nonzero polynomial by \((x^2 + 5x + 7)\) increases the degree by 2, and then differentiating twice reduces the degree by 2. Thus \(T\) is indeed an operator on \(\mathcal{P}_m(\mathbb{R})\).

Every polynomial whose second derivative equals 0 is of the form \(ax + b\), where \(a, b \in \mathbb{R}\). Thus \(\text{null } T = \{0\}\). Hence \(T\) is injective.

Now 3.69 implies that \(T\) is surjective. Thus there exists a polynomial \(p \in \mathcal{P}_m(\mathbb{R})\) such that \(((x^2 + 5x + 7)p)'' = q\), as desired.

Exercise 30 in Section 6.A gives a similar but more spectacular application of 3.69. The result in that exercise is quite difficult to prove without using linear algebra.

**EXERCISES 3.D**

1. Suppose \(T \in \mathcal{L}(U, V)\) and \(S \in \mathcal{L}(V, W)\) are both invertible linear maps. Prove that \(ST \in \mathcal{L}(U, W)\) is invertible and that \((ST)^{-1} = T^{-1}S^{-1}\).

2. Suppose \(V\) is finite-dimensional and \(\text{dim } V > 1\). Prove that the set of noninvertible operators on \(V\) is not a subspace of \(\mathcal{L}(V)\).

3. Suppose \(V\) is finite-dimensional, \(U\) is a subspace of \(V\), and \(S \in \mathcal{L}(U, V)\). Prove there exists an invertible operator \(T \in \mathcal{L}(V)\) such that \(Tu = Su\) for every \(u \in U\) if and only if \(S\) is injective.

4. Suppose \(W\) is finite-dimensional and \(T_1, T_2 \in \mathcal{L}(V, W)\). Prove that \(\text{null } T_1 = \text{null } T_2\) if and only if there exists an invertible operator \(S \in \mathcal{L}(W)\) such that \(T_1 = ST_2\).

5. Suppose \(V\) is finite-dimensional and \(T_1, T_2 \in \mathcal{L}(V, W)\). Prove that \(\text{range } T_1 = \text{range } T_2\) if and only if there exists an invertible operator \(S \in \mathcal{L}(V)\) such that \(T_1 = T_2S\).

6. Suppose \(V\) and \(W\) are finite-dimensional and \(T_1, T_2 \in \mathcal{L}(V, W)\). Prove that there exist invertible operators \(R \in \mathcal{L}(V)\) and \(S \in \mathcal{L}(W)\) such that \(T_1 = ST_2R\) if and only if \(\text{dim } \text{null } T_1 = \text{dim } \text{null } T_2\).

7. Suppose \(V\) and \(W\) are finite-dimensional. Let \(v \in V\). Let \(E = \{T \in \mathcal{L}(V, W) : Tv = 0\}\).

(a) Show that \(E\) is a subspace of \(\mathcal{L}(V, W)\).

(b) Suppose \(v \neq 0\). What is \(\text{dim } E\)?
8 Suppose $V$ is finite-dimensional and $T : V \to W$ is a surjective linear map of $V$ onto $W$. Prove that there is a subspace $U$ of $V$ such that $T|_U$ is an isomorphism of $U$ onto $W$. (Here $T|_U$ means the function $T$ restricted to $U$. In other words, $T|_U$ is the function whose domain is $U$, with $T|_U(u) = Tu$ for every $u \in U$.)

9 Suppose $V$ is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST$ is invertible if and only if both $S$ and $T$ are invertible.

10 Suppose $V$ is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

11 Suppose $V$ is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that $T$ is invertible and that $T^{-1} = US$.

12 Show that the result in the previous exercise can fail without the hypothesis that $V$ is finite-dimensional.

13 Suppose $V$ is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that $RST$ is surjective. Prove that $S$ is injective.

14 Suppose $v_1, \ldots, v_n$ is a basis of $V$. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by

$$Tv = M(v)$$

is an isomorphism of $V$ onto $\mathbb{F}^{n,1}$; here $M(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_n$.

15 Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an $m$-by-$n$ matrix $A$ such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$.

16 Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that $T$ is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

17 Suppose $V$ is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$.

18 Show that $V$ and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

19 Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that $T$ is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

(a) Prove that $T$ is surjective.

(b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$. 
2 Suppose $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that $V_j$ is finite-dimensional for each $j = 1, \ldots, m$.

3 Give an example of a vector space $V$ and subspaces $U_1, U_2$ of $V$ such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Hint: The vector space $V$ must be infinite-dimensional.

4 Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

5 Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

6 For $n$ a positive integer, define $V^n$ by

$$V^n = \frac{V \times \cdots \times V}{n \text{ times}}.$$ 

Prove that $V^n$ and $\mathcal{L}(F^n, V)$ are isomorphic vector spaces.

7 Suppose $v, x$ are vectors in $V$ and $U, W$ are subspaces of $V$ such that $v + U = x + W$. Prove that $U = W$.

8 Prove that a nonempty subset $A$ of $V$ is an affine subset of $V$ if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in F$.

9 Suppose $A_1$ and $A_2$ are affine subsets of $V$. Prove that the intersection $A_1 \cap A_2$ is either an affine subset of $V$ or the empty set.

10 Prove that the intersection of every collection of affine subsets of $V$ is either an affine subset of $V$ or the empty set.

11 Suppose $v_1, \ldots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, \ldots, \lambda_m \in F \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}.$$ 

(a) Prove that $A$ is an affine subset of $V$.

(b) Prove that every affine subset of $V$ that contains $v_1, \ldots, v_m$ also contains $A$.

(c) Prove that $A = v + U$ for some $v \in V$ and some subspace $U$ of $V$ with $\text{dim } U \leq m - 1$.

12 Suppose $U$ is a subspace of $V$ such that $V/U$ is finite-dimensional. Prove that $V$ is isomorphic to $U \times (V/U)$. 

0
Suppose \( U \) is a subspace of \( V \) and \( v_1 + U, \ldots, v_m + U \) is a basis of \( V/U \) and \( u_1, \ldots, u_n \) is a basis of \( U \). Prove that \( v_1, \ldots, v_m, u_1, \ldots, u_n \) is a basis of \( V \).

(a) Show that \( U \) is a subspace of \( F^\infty \).

(b) Prove that \( F^\infty / U \) is infinite-dimensional.

Suppose \( \varphi \in \mathcal{L}(V, F) \) and \( \varphi \neq 0 \). Prove that \( \dim V/(\text{null } \varphi) = 1 \).

Suppose \( U \) is a subspace of \( V \) such that \( \dim V/U = 1 \). Prove that there exists \( \varphi \in \mathcal{L}(V, F) \) such that \( \text{null } \varphi = U \).

Suppose \( U \) is a subspace of \( V \) such that \( \dim V/U = 1 \). Prove that there exists a subspace \( W \) of \( V \) such that \( \dim W = \dim V/U \) and \( V = U \oplus W \).

Suppose \( T \in \mathcal{L}(V, W) \) and \( U \) is a subspace of \( V \). Let \( \pi \) denote the quotient map from \( V \) onto \( V/U \). Prove that there exists \( S \in \mathcal{L}(V/U, W) \) such that \( T = S \circ \pi \) if and only if \( U \subset \text{null } T \).

Find a correct statement analogous to 3.78 that is applicable to finite sets, with unions analogous to sums of subspaces and disjoint unions analogous to direct sums.

Suppose \( U \) is a subspace of \( V \). Define \( \Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W) \) by \( \Gamma(S) = S \circ \pi \).

(a) Show that \( \Gamma \) is a linear map.

(b) Show that \( \Gamma \) is injective.

(c) Show that \( \text{range } \Gamma = \{ T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U \} \).
HWK 9
for discussion next two weeks relevant to $E_4$. 
EXERCISES 3.F

1. Explain why every linear functional is either surjective or the zero map.

2. Give three distinct examples of linear functionals on \( \mathbb{R}^{[0,1]} \).

3. Suppose \( V \) is finite-dimensional and \( v \in V \) with \( v \neq 0 \). Prove that there exists \( \varphi \in V' \) such that \( \varphi(v) = 1 \).

4. Suppose \( V \) is finite-dimensional and \( U \) is a subspace of \( V \) such that \( U \neq V \). Prove that there exists \( \varphi \in V' \) such that \( \varphi(u) = 0 \) for every \( u \in U \) but \( \varphi \neq 0 \).

5. Suppose \( V_1, \ldots, V_m \) are vector spaces. Prove that \( (V_1 \times \cdots \times V_m)' \) and \( V'_1 \times \cdots \times V'_m \) are isomorphic vector spaces.

6. Suppose \( V \) is finite-dimensional and \( v_1, \ldots, v_m \in V \). Define a linear map \( \Gamma : V' \to \mathbb{F}^m \) by

\[
\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)).
\]

(a) Prove that \( v_1, \ldots, v_m \) spans \( V \) if and only if \( \Gamma \) is injective.

(b) Prove that \( v_1, \ldots, v_m \) is linearly independent if and only if \( \Gamma \) is surjective.

7. Suppose \( m \) is a positive integer. Show that the dual basis of the basis \( 1, x, \ldots, x^m \) of \( \mathcal{P}_m(\mathbb{R}) \) is \( \varphi_0, \varphi_1, \ldots, \varphi_m \), where \( \varphi_j(p) = \frac{p^{(j)}(0)}{j!} \). Here \( p^{(j)} \) denotes the \( j \)th derivative of \( p \), with the understanding that the 0th derivative of \( p \) is \( p \).

8. Suppose \( m \) is a positive integer.

(a) Show that \( 1, x - 5, \ldots, (x - 5)^m \) is a basis of \( \mathcal{P}_m(\mathbb{R}) \).

(b) What is the dual basis of the basis in part (a)?

9. Suppose \( v_1, \ldots, v_n \) is a basis of \( V \) and \( \varphi_1, \ldots, \varphi_n \) is the corresponding dual basis of \( V' \). Suppose \( \psi \in V' \). Prove that

\[
\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.
\]

10. Prove the first two bullet points in 3.101.
Suppose $A$ is an $m$-by-$n$ matrix with $A \neq 0$. Prove that the rank of $A$ is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbb{F}^m$ and $(d_1, \ldots, d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

Show that the dual map of the identity map on $V$ is the identity map on $V^\prime$.

Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Suppose $\varphi_1, \varphi_2$ denotes the dual basis of the standard basis of $\mathbb{R}^2$ and $\psi_1, \psi_2, \psi_3$ denotes the dual basis of the standard basis of $\mathbb{R}^3$.

(a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of $\psi_1, \psi_2, \psi_3$.

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in \mathbb{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbb{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = \int_0^1 p(x) \, dx$. Evaluate $(T'(\varphi))(x^3)$.

Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Suppose $V$ and $W$ are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Suppose $U \subset V$. Explain why $U^0 = \{ \varphi \in V' : U \subset \text{null } \varphi \}$.

Suppose $V$ is finite-dimensional and $U \subset V$. Show that $U = \{0\}$ if and only if $U^0 = V'$.

Suppose $V$ is finite-dimensional and $U$ is a subspace of $V$. Show that $U = V$ if and only if $U^0 = \{0\}$.

Suppose $U$ and $W$ are subsets of $V$ with $U \subset W$. Prove that $W^0 \subset U^0$.

Suppose $V$ is finite-dimensional and $U$ and $W$ are subspaces of $V$ with $W^0 \subset U^0$. Prove that $U \subset W$.

Suppose $U, W$ are subspaces of $V$. Show that $(U + W)^0 = U^0 \cap W^0$. 

Show that the dual map of the identity map on $V$ is the identity map on $V^\prime$.

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in \mathbb{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbb{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = \int_0^1 p(x) \, dx$. Evaluate $(T'(\varphi))(x^3)$.
23 Suppose $V$ is finite-dimensional and $U$ and $W$ are subspaces of $V$. Prove that $(U \cap W)^0 = U^0 + W^0$.

24 Prove 3.106 using the ideas sketched in the discussion before the statement of 3.106.

25 Suppose $V$ is finite-dimensional and $U$ is a subspace of $V$. Show that $U = \{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0 \}$.

26 Suppose $V$ is finite-dimensional and $\Gamma$ is a subspace of $V'$. Show that $\Gamma = \{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Gamma \}^0$.

27 Suppose $T \in \mathcal{L}(P_5(\mathbb{R}), P_5(\mathbb{R}))$ and null $T' = \text{span}(\varphi)$, where $\varphi$ is the linear functional on $P_5(\mathbb{R})$ defined by $\varphi(p) = p(8)$. Prove that range $T = \{ p \in P_5(\mathbb{R}) : p(8) = 0 \}$.

28 Suppose $V$ and $W$ are finite-dimensional, $T \in \mathcal{L}(V, W)$, and there exists $\varphi \in W'$ such that null $T' = \text{span}(\varphi)$. Prove that range $T = \text{null } \varphi$.

29 Suppose $V$ and $W$ are finite-dimensional, $T \in \mathcal{L}(V, W)$, and there exists $\varphi \in V'$ such that range $T' = \text{span}(\varphi)$. Prove that null $T = \text{null } \varphi$.

30 Suppose $V$ is finite-dimensional and $\varphi_1, \ldots, \varphi_m$ is a linearly independent list in $V'$. Prove that $\dim ((\text{null } \varphi_1) \cap \cdots \cap (\text{null } \varphi_m)) = (\dim V) - m$.

31 Suppose $V$ is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of $V'$. Show that there exists a basis of $V$ whose dual basis is $\varphi_1, \ldots, \varphi_n$.

32 Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of $V$. Prove that the following are equivalent:

(a) $T$ is invertible.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$. 
Suppose $m$ and $n$ are positive integers. Prove that the function that takes $A$ to $A^t$ is a linear map from $F^{m,n}$ to $F^{n,m}$. Furthermore, prove that this linear map is invertible.

The double dual space of $V$, denoted $V''$, is defined to be the dual space of $V'$. In other words, $V'' = (V')'$. Define $\Lambda : V \rightarrow V''$ by

$$\Lambda(v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

(a) Show that $\Lambda$ is a linear map from $V$ to $V''$.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if $V$ is finite-dimensional, then $\Lambda$ is an isomorphism from $V$ onto $V''$.

[Suppose $V$ is finite-dimensional. Then $V$ and $V'$ are isomorphic, but finding an isomorphism from $V$ onto $V'$ generally requires choosing a basis of $V$. In contrast, the isomorphism $\Lambda$ from $V$ onto $V''$ does not require a choice of basis and thus is considered more natural.]

Show that $(\mathcal{P}(\mathbb{R}))'$ and $\mathbb{R}^\infty$ are isomorphic.

Suppose $U$ is a subspace of $V$. Let $i : U \rightarrow V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that null $i' = U^0$.

(b) Prove that if $V$ is finite-dimensional, then range $i' = U'$.

(c) Conclude that $i'$ is an isomorphism from $V'/U^0$ onto $U'$.

[The isomorphism in part (c) is natural in that it does not depend on a choice of basis in either vector space.]

Suppose $U$ is a subspace of $V$. Let $\pi : V \rightarrow V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}(V/U)', V')$.

(a) Show that $\pi'$ is injective.

(b) Show that range $\pi' = U^0$.

(c) Conclude that $\pi'$ is an isomorphism from $(V/U)'$ onto $U^0$.

[The isomorphism in part (c) is natural in that it does not depend on a choice of basis in either vector space. In fact, there is no assumption here that any of these vector spaces are finite-dimensional.]
Husk 10 on chip 5

due to grade Nov 13

2 weeks
of work.
3 Suppose \( S, T \in \mathcal{L}(V) \) are such that \( ST = TS \). Prove that range \( S \) is invariant under \( T \).

4 Suppose that \( T \in \mathcal{L}(V) \) and \( U_1, \ldots, U_m \) are subspaces of \( V \) invariant under \( T \). Prove that \( U_1 + \cdots + U_m \) is invariant under \( T \).

5 Suppose \( T \in \mathcal{L}(V) \). Prove that the intersection of every collection of subspaces of \( V \) invariant under \( T \) is invariant under \( T \).

6 Prove or give a counterexample: if \( V \) is finite-dimensional and \( U \) is a subspace of \( V \) that is invariant under every operator on \( V \), then \( U = \{0\} \) or \( U = V \).

7 Suppose \( T \in \mathcal{L}(\mathbb{R}^2) \) is defined by \( T(x, y) = (-3y, x) \). Find the eigenvalues of \( T \).

8 Define \( T \in \mathcal{L}(\mathbb{F}^2) \) by
\[
T(w, z) = (z, w).
\]
Find all eigenvalues and eigenvectors of \( T \).

9 Define \( T \in \mathcal{L}(\mathbb{F}^3) \) by
\[
T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).
\]
Find all eigenvalues and eigenvectors of \( T \).

10 Define \( T \in \mathcal{L}(\mathbb{F}^n) \) by
\[
T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n).
\]
(a) Find all eigenvalues and eigenvectors of \( T \).
(b) Find all invariant subspaces of \( T \).

11 Define \( T : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R}) \) by \( Tp = p' \). Find all eigenvalues and eigenvectors of \( T \).

12 Define \( T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R})) \) by
\[
(Tp)(x) = xp'(x)
\]
for all \( x \in \mathbb{R} \). Find all eigenvalues and eigenvectors of \( T \).

13 Suppose \( V \) is finite-dimensional, \( T \in \mathcal{L}(V) \), and \( \lambda \in \mathbb{F} \). Prove that there exists \( \alpha \in \mathbb{F} \) such that \( |\alpha - \lambda| < \frac{1}{1000} \) and \( T - \alpha I \) is invertible.
14 Suppose $V = U \oplus W$, where $U$ and $W$ are nonzero subspaces of $V$. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of $P$.

15 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that $T$ and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of $T$ and the eigenvectors of $S^{-1}TS$?

16 Suppose $V$ is a complex vector space, $T \in \mathcal{L}(V)$, and the matrix of $T$ with respect to some basis of $V$ contains only real entries. Show that if $\lambda$ is an eigenvalue of $T$, then so is $\bar{\lambda}$.

17 Give an example of an operator $T \in \mathcal{L}(\mathbb{R}^4)$ such that $T$ has no (real) eigenvalues.

18 Show that the operator $T \in \mathcal{L}(\mathbb{C}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

19 Suppose $n$ is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n);$$

in other words, $T$ is the operator whose matrix (with respect to the standard basis) consists of all 1’s. Find all eigenvalues and eigenvectors of $T$.

20 Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots).$$

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that $\lambda$ is an eigenvalue of $T$ if and only if $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

(b) Prove that $T$ and $T^{-1}$ have the same eigenvectors.
Suppose $\lambda_1, \ldots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

*Hint:* Let $V = \text{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$, and define an operator $T \in \mathcal{L}(V)$ by $Tf = f'$. Find eigenvalues and eigenvectors of $T$.

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{range } T) = 0$.

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null } T)$ is injective if and only if $(\text{null } T) \cap (\text{range } T) = \{0\}$.

Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $U$ is invariant under $T$. Prove that each eigenvalue of $T/U$ is an eigenvalue of $T$.

*The exercise below asks you to verify that the hypothesis that $V$ is finite-dimensional is needed for the exercise above.*

Give an example of a vector space $V$, an operator $T \in \mathcal{L}(V)$, and a subspace $U$ of $V$ that is invariant under $T$ such that $T/U$ has an eigenvalue that is not an eigenvalue of $T$. 
EXERCISES 5.B

1. Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer $n$ such that $T^n = 0$.
   (a) Prove that $I - T$ is invertible and that
   $$(I - T)^{-1} = I + T + \cdots + T^{n-1}.$$
   (b) Explain how you would guess the formula above.

2. Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose $\lambda$ is an eigenvalue of $T$. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

3. Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and $-1$ is not an eigenvalue of $T$. Prove that $T = I$.

4. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \text{null } P \oplus \text{range } P$.

5. Suppose $S, T \in \mathcal{L}(V)$ and $S$ is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that
   $$pSTS^{-1} = Sp(T)S^{-1}.$$

6. Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$ invariant under $T$. Prove that $U$ is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$.

7. Suppose $T \in \mathcal{L}(V)$. Prove that $9$ is an eigenvalue of $T^2$ if and only if $3$ or $-3$ is an eigenvalue of $T$.

8. Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -1$.

9. Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let $p$ be a nonzero polynomial of smallest degree such that $p(T)v = 0$. Prove that every zero of $p$ is an eigenvalue of $T$.

10. Suppose $T \in \mathcal{L}(V)$ and $v$ is an eigenvector of $T$ with eigenvalue $\lambda$. Suppose $p \in \mathcal{P}(\mathbb{F})$. Prove that $p(T)v = p(\lambda)v$.

11. Suppose $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{C})$ is a polynomial, and $\alpha \in \mathbb{C}$. Prove that $\alpha$ is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue $\lambda$ of $T$.

12. Show that the result in the previous exercise does not hold if $\mathbb{C}$ is replaced with $\mathbb{R}$.
The converse of 5.44 is not true. For example, the operator $T$ defined on the three-dimensional space $\mathbb{F}^3$ by

$$T(z_1, z_2, z_3) = (4z_1, 4z_2, 5z_3)$$

has only two eigenvalues (4 and 5), but this operator has a diagonal matrix with respect to the standard basis.

In later chapters we will find additional conditions that imply that certain operators are diagonalizable.

**EXERCISES 5.C**

1. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \text{null } T \oplus \text{range } T$.

2. Prove the converse of the statement in the exercise above or give a counterexample to the converse.

3. Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:
   
   (a) $V = \text{null } T \oplus \text{range } T$.
   
   (b) $V = \text{null } T + \text{range } T$.
   
   (c) $\text{null } T \cap \text{range } T = \{0\}$.

4. Give an example to show that the exercise above is false without the hypothesis that $V$ is finite-dimensional.

5. Suppose $V$ is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that $T$ is diagonalizable if and only if

$$V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$$

for every $\lambda \in \mathbb{C}$.

6. Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as $T$ (not necessarily with the same eigenvalues). Prove that $ST = TS$.

7. Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix $A$ with respect to some basis of $V$ and that $\lambda \in \mathbb{F}$. Prove that $\lambda$ appears on the diagonal of $A$ precisely $\dim E(\lambda, T)$ times.

8. Suppose $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.
9 Suppose \( T \in \mathcal{L}(V) \) is invertible. Prove that \( E(\lambda, T) = E\left(\frac{1}{\lambda}, T^{-1}\right) \) for every \( \lambda \in \mathbb{F} \) with \( \lambda \neq 0 \).

10 Suppose that \( V \) is finite-dimensional and \( T \in \mathcal{L}(V) \). Let \( \lambda_1, \ldots, \lambda_m \) denote the distinct nonzero eigenvalues of \( T \). Prove that
\[
\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \text{range } T.
\]

11 Verify the assertion in Example 5.40.

12 Suppose \( R, T \in \mathcal{L}(\mathbb{F}^3) \) each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator \( S \in \mathcal{L}(\mathbb{F}^3) \) such that \( R = S^{-1}TS \).

13 Find \( R, T \in \mathcal{L}(\mathbb{F}^4) \) such that \( R \) and \( T \) each have 2, 6, 7 as eigenvalues, \( R \) and \( T \) have no other eigenvalues, and there does not exist an invertible operator \( S \in \mathcal{L}(\mathbb{F}^4) \) such that \( R = S^{-1}TS \).

14 Find \( T \in \mathcal{L}(\mathbb{C}^3) \) such that 6 and 7 are eigenvalues of \( T \) and such that \( T \) does not have a diagonal matrix with respect to any basis of \( \mathbb{C}^3 \).

15 Suppose \( T \in \mathcal{L}(\mathbb{C}^3) \) is such that 6 and 7 are eigenvalues of \( T \). Furthermore, suppose \( T \) does not have a diagonal matrix with respect to any basis of \( \mathbb{C}^3 \). Prove that there exists \( (x, y, z) \in \mathbb{F}^3 \) such that \( T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z) \).

16 The **Fibonacci sequence** \( F_1, F_2, \ldots \) is defined by
\[
F_1 = 1, \ F_2 = 1, \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3.
\]
Define \( T \in \mathcal{L}(\mathbb{R}^2) \) by \( T(x, y) = (y, x + y) \).

(a) Show that \( T^n(0, 1) = (F_n, F_{n+1}) \) for each positive integer \( n \).
(b) Find the eigenvalues of \( T \).
(c) Find a basis of \( \mathbb{R}^2 \) consisting of eigenvectors of \( T \).
(d) Use the solution to part (c) to compute \( T^n(0, 1) \). Conclude that
\[
F_n = \frac{1}{\sqrt{5}}\left[ \left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n \right]
\]
for each positive integer \( n \).
(e) Use part (d) to conclude that for each positive integer \( n \), the Fibonacci number \( F_n \) is the integer that is closest to
\[
\frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n.
\]
Hwk 11 due to grades Nov 27 relevant to Exam 5 on Dec 4.
EXERCISES 6.A

1. Show that the function that takes \((x_1, x_2), (y_1, y_2)\) \(\in \mathbb{R}^2 \times \mathbb{R}^2\) to \(|x_1 y_1| + |x_2 y_2|\) is not an inner product on \(\mathbb{R}^2\).

2. Show that the function that takes \((x_1, x_2, x_3), (y_1, y_2, y_3)\) \(\in \mathbb{R}^3 \times \mathbb{R}^3\) to \(x_1 y_1 + x_3 y_3\) is not an inner product on \(\mathbb{R}^3\).

3. Suppose \(F = \mathbb{R}\) and \(V \neq \{0\}\). Replace the positivity condition (which states that \(\langle v, v \rangle \geq 0\) for all \(v \in V\)) in the definition of an inner product (6.3) with the condition that \(\langle v, v \rangle > 0\) for some \(v \in V\). Show that this change in the definition does not change the set of functions from \(V \times V\) to \(\mathbb{R}\) that are inner products on \(V\).

4. Suppose \(V\) is a real inner product space.
   (a) Show that \(\langle u + v, u - v \rangle = \|u\|^2 - \|v\|^2\) for every \(u, v \in V\).
   (b) Show that if \(u, v \in V\) have the same norm, then \(u + v\) is orthogonal to \(u - v\).
   (c) Use part (b) to show that the diagonals of a rhombus are perpendicular to each other.

5. Suppose \(T \in \mathcal{L}(V)\) is such that \(\|Tv\| \leq \|v\|\) for every \(v \in V\). Prove that \(T - \sqrt{2}I\) is invertible.

6. Suppose \(u, v \in V\). Prove that \(\langle u, v \rangle = 0\) if and only if
   \[\|u\| \leq \|u + av\|\]
   for all \(a \in F\).

7. Suppose \(u, v \in V\). Prove that \(\|au + bv\| = \|bu + av\|\) for all \(a, b \in \mathbb{R}\) if and only if \(\|u\| = \|v\|\).

8. Suppose \(u, v \in V\) and \(\|u\| = \|v\| = 1\) and \(\langle u, v \rangle = 1\). Prove that \(u = v\).

9. Suppose \(u, v \in V\) and \(\|u\| \leq 1\) and \(\|v\| \leq 1\). Prove that
   \[\sqrt{1 - \|u\|^2} \sqrt{1 - \|v\|^2} \leq 1 - |\langle u, v \rangle|\].

10. Find vectors \(u, v \in \mathbb{R}^2\) such that \(u\) is a scalar multiple of \((1, 3)\), \(v\) is orthogonal to \((1, 3)\), and \((1, 2) = u + v\).
11. Prove that
\[ 16 \leq (a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \]
for all positive numbers \( a, b, c, d \).

12. Prove that
\[ (x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2) \]
for all positive integers \( n \) and all real numbers \( x_1, \ldots, x_n \).

13. Suppose \( u, v \) are nonzero vectors in \( \mathbb{R}^2 \). Prove that
\[ \langle u, v \rangle = \|u\|\|v\| \cos \theta, \]
where \( \theta \) is the angle between \( u \) and \( v \) (thinking of \( u \) and \( v \) as arrows with
initial point at the origin).

*Hint:* Draw the triangle formed by \( u, v, \) and \( u - v \); then use the law of cosines.

14. The angle between two vectors (thought of as arrows with initial point at
the origin) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) can be defined geometrically. However, geometry
is not as clear in \( \mathbb{R}^n \) for \( n > 3 \). Thus the angle between two nonzero
vectors \( x, y \in \mathbb{R}^n \) is defined to be
\[ \arccos \frac{\langle x, y \rangle}{\|x\|\|y\|}, \]
where the motivation for this definition comes from the previous exercise.
Explain why the Cauchy–Schwarz Inequality is needed to show that this
definition makes sense.

15. Prove that
\[ \left( \sum_{j=1}^{n} a_j b_j \right)^2 \leq \left( \sum_{j=1}^{n} j a_j^2 \right) \left( \sum_{j=1}^{n} b_j^2 / j \right) \]
for all real numbers \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \).

16. Suppose \( u, v \in V \) are such that
\[ \|u\| = 3, \quad \|u + v\| = 4, \quad \|u - v\| = 6. \]
What number does \( \|v\| \) equal?
17 Prove or disprove: there is an inner product on \( \mathbb{R}^2 \) such that the associated norm is given by
\[
\| (x, y) \| = \max\{x, y\}
\]
for all \((x, y) \in \mathbb{R}^2\).

18 Suppose \( p > 0 \). Prove that there is an inner product on \( \mathbb{R}^2 \) such that the associated norm is given by
\[
\| (x, y) \| = (x^p + y^p)^{1/p}
\]
for all \((x, y) \in \mathbb{R}^2\) if and only if \( p = 2 \).

19 Suppose \( V \) is a real inner product space. Prove that
\[
\langle u, v \rangle = \frac{\| u + v \|^2 - \| u - v \|^2}{4}
\]
for all \( u, v \in V \).

20 Suppose \( V \) is a complex inner product space. Prove that
\[
\langle u, v \rangle = \frac{\| u + v \|^2 - \| u - v \|^2 + \| u + i v \|^2 - \| u - i v \|^2}{4}
\]
for all \( u, v \in V \).

21 A norm on a vector space \( U \) is a function \( \| \| : U \to [0, \infty) \) such that \( \| u \| = 0 \) if and only if \( u = 0 \), \( \| \alpha u \| = |\alpha| \| u \| \) for all \( \alpha \in \mathbb{F} \) and all \( u \in U \), and \( \| u + v \| \leq \| u \| + \| v \| \) for all \( u, v \in U \). Prove that a norm satisfying the parallelogram equality comes from an inner product (in other words, show that if \( \| \| \) is a norm on \( U \) satisfying the parallelogram equality, then there is an inner product \( \langle , \rangle \) on \( U \) such that \( \| u \| = \langle u, u \rangle^{1/2} \) for all \( u \in U \)).

22 Show that the square of an average is less than or equal to the average of the squares. More precisely, show that if \( a_1, \ldots, a_n \in \mathbb{R} \), then the square of the average of \( a_1, \ldots, a_n \) is less than or equal to the average of \( a_1^2, \ldots, a_n^2 \).

23 Suppose \( V_1, \ldots, V_m \) are inner product spaces. Show that the equation
\[
\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle
\]
defines an inner product on \( V_1 \times \cdots \times V_m \).

[In the expression above on the right, \( \langle u_1, v_1 \rangle \) denotes the inner product on \( V_1 \), \ldots, \( \langle u_m, v_m \rangle \) denotes the inner product on \( V_m \). Each of the spaces \( V_1, \ldots, V_m \) may have a different inner product, even though the same notation is used here.]
Solution  Let \( \varphi(p) = \int_{-1}^{1} p(t) \left( \cos(\pi t) \right) dt \). Applying formula 6.43 from the proof above, and using the orthonormal basis from Example 6.33, we have

\[
u(x) = \left( \int_{-1}^{1} \sqrt{\frac{1}{2}} \left( \cos(\pi t) \right) dt \right) \sqrt{\frac{1}{2}} + \left( \int_{-1}^{1} \sqrt{\frac{3}{2}} t \left( \cos(\pi t) \right) dt \right) \sqrt{\frac{3}{2}} x
+ \left( \int_{-1}^{1} \sqrt{\frac{45}{8}} \left( t^2 - \frac{1}{3} \right) \left( \cos(\pi t) \right) dt \right) \sqrt{\frac{45}{8}} \left( x^2 - \frac{1}{3} \right).
\]

A bit of calculus shows that

\[
u(x) = -\frac{45}{2\pi^2} \left( x^2 - \frac{1}{3} \right).
\]

Suppose \( V \) is finite-dimensional and \( \varphi \) a linear functional on \( V \). Then 6.43 gives a formula for the vector \( u \) that satisfies \( \varphi(v) = \langle v, u \rangle \) for all \( v \in V \). Specifically, we have

\[
u = \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n.
\]

The right side of the equation above seems to depend on the orthonormal basis \( e_1, \ldots, e_n \) as well as on \( \varphi \). However, 6.42 tells us that \( u \) is uniquely determined by \( \varphi \). Thus the right side of the equation above is the same regardless of which orthonormal basis \( e_1, \ldots, e_n \) of \( V \) is chosen.

EXERCISES 6.B

1. (a) Suppose \( \theta \in \mathbb{R} \). Show that \( (\cos \theta, \sin \theta), (-\sin \theta, \cos \theta) \) and \( (\cos \theta, \sin \theta), (\sin \theta, -\cos \theta) \) are orthonormal bases of \( \mathbb{R}^2 \).

(b) Show that each orthonormal basis of \( \mathbb{R}^2 \) is of the form given by one of the two possibilities of part (a).

2. Suppose \( e_1, \ldots, e_m \) is an orthonormal list of vectors in \( V \). Let \( v \in V \). Prove that

\[
\| v \|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2
\]

if and only if \( v \in \text{span}(e_1, \ldots, e_m) \).

3. Suppose \( T \in \mathcal{L}(\mathbb{R}^3) \) has an upper-triangular matrix with respect to the basis \( (1, 0, 0), (1, 1, 1), (1, 1, 2) \). Find an orthonormal basis of \( \mathbb{R}^3 \) (use the usual inner product on \( \mathbb{R}^3 \)) with respect to which \( T \) has an upper-triangular matrix.
Suppose $F = \mathbb{C}$, $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, all the eigenvalues of $T$ have absolute value less than 1, and $\epsilon > 0$. Prove that there exists a positive integer $m$ such that $\|T^m v\| \leq \epsilon \|v\|$ for every $v \in V$.

For $u \in V$, let $\Phi u$ denote the linear functional on $V$ defined by

$$(\Phi u)(v) = \langle v, u \rangle$$

for $v \in V$.

(a) Show that if $F = \mathbb{R}$, then $\Phi$ is a linear map from $V$ to $V'$. (Recall from Section 3.F that $V' = \mathcal{L}(V, F)$ and that $V'$ is called the dual space of $V$.)

(b) Show that if $F = \mathbb{C}$ and $V \neq \{0\}$, then $\Phi$ is not a linear map.

(c) Show that $\Phi$ is injective.

(d) Suppose $F = \mathbb{R}$ and $V$ is finite-dimensional. Use parts (a) and (c) and a dimension-counting argument (but without using 6.42) to show that $\Phi$ is an isomorphism from $V$ onto $V'$.

[Part (d) gives an alternative proof of the Riesz Representation Theorem (6.42) when $F = \mathbb{R}$. Part (d) also gives a natural isomorphism (meaning that it does not depend on a choice of basis) from a finite-dimensional real inner product space onto its dual space.]
1. Suppose \( v_1, \ldots, v_m \in V \). Prove that
\[
\{v_1, \ldots, v_m\}^\perp = (\text{span}(v_1, \ldots, v_m))^\perp.
\]

2. Suppose \( U \) is a finite-dimensional subspace of \( V \). Prove that \( U^\perp = \{0\} \) if and only if \( U = V \).

[Exercise 14(a) shows that the result above is not true without the hypothesis that \( U \) is finite-dimensional.]

3. Suppose \( U \) is a subspace of \( V \) with basis \( u_1, \ldots, u_m \) and
\[
u_1, \ldots, u_m, w_1, \ldots, w_n
\]
is a basis of \( V \). Prove that if the Gram–Schmidt Procedure is applied to the basis of \( V \) above, producing a list \( e_1, \ldots, e_m, f_1, \ldots, f_n \), then \( e_1, \ldots, e_m \) is an orthonormal basis of \( U \) and \( f_1, \ldots, f_n \) is an orthonormal basis of \( U^\perp \).

4. Suppose \( U \) is the subspace of \( \mathbf{R}^4 \) defined by
\[
U = \text{span}((1, 2, 3, -4), (-5, 4, 3, 2)).
\]
Find an orthonormal basis of \( U \) and an orthonormal basis of \( U^\perp \).

5. Suppose \( V \) is finite-dimensional and \( U \) is a subspace of \( V \). Show that \( P_{U^\perp} = I - P_U \), where \( I \) is the identity operator on \( V \).

6. Suppose \( U \) and \( W \) are finite-dimensional subspaces of \( V \). Prove that \( P_U P_W = 0 \) if and only if \( \langle u, w \rangle = 0 \) for all \( u \in U \) and all \( w \in W \).

7. Suppose \( V \) is finite-dimensional and \( P \in \mathcal{L}(V) \) is such that \( P^2 = P \) and every vector in null \( P \) is orthogonal to every vector in range \( P \). Prove that there exists a subspace \( U \) of \( V \) such that \( P = P_U \).

8. Suppose \( V \) is finite-dimensional and \( P \in \mathcal{L}(V) \) is such that \( P^2 = P \) and
\[
\|Pv\| \leq \|v\|
\]
for every \( v \in V \). Prove that there exists a subspace \( U \) of \( V \) such that \( P = P_U \).

9. Suppose \( T \in \mathcal{L}(V) \) and \( U \) is a finite-dimensional subspace of \( V \). Prove that \( U \) is invariant under \( T \) if and only if \( P_U TP_U = TP_U \).
HWK 12 Due to grade Dec 4.

E5 on Dec 4 will be based on chps 5, 6, 7 (not including hwk 12).
7.22 Orthogonal eigenvectors for normal operators

Suppose \( T \in \mathcal{L}(V) \) is normal. Then eigenvectors of \( T \) corresponding to distinct eigenvalues are orthogonal.

Proof Suppose \( \alpha, \beta \) are distinct eigenvalues of \( T \), with corresponding eigenvectors \( u, v \). Thus \( Tu = \alpha u \) and \( Tv = \beta v \). From 7.21 we have \( T^*v = \overline{\beta}v \). Thus

\[
(\alpha - \beta) \langle u, v \rangle = \langle \alpha u, v \rangle - \langle u, \overline{\beta}v \rangle = \langle Tu, v \rangle - \langle u, T^*v \rangle = 0.
\]

Because \( \alpha \neq \beta \), the equation above implies that \( \langle u, v \rangle = 0 \). Thus \( u \) and \( v \) are orthogonal, as desired.

EXERCISES 7.A

1. Suppose \( n \) is a positive integer. Define \( T \in \mathcal{L}(\mathbb{F}^n) \) by

\[
T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).
\]

Find a formula for \( T^*(z_1, \ldots, z_n) \).

2. Suppose \( T \in \mathcal{L}(V) \) and \( \lambda \in \mathbb{F} \). Prove that \( \lambda \) is an eigenvalue of \( T \) if and only if \( \overline{\lambda} \) is an eigenvalue of \( T^* \).

3. Suppose \( T \in \mathcal{L}(V) \) and \( U \) is a subspace of \( V \). Prove that \( U \) is invariant under \( T \) if and only if \( U^\perp \) is invariant under \( T^* \).

4. Suppose \( T \in \mathcal{L}(V, W) \). Prove that

(a) \( T \) is injective if and only if \( T^* \) is surjective;
(b) \( T \) is surjective if and only if \( T^* \) is injective.

5. Prove that

\[
\dim \ker T^* = \dim \ker T + \dim W - \dim V
\]

and

\[
\dim \text{range } T^* = \dim \text{range } T
\]

for every \( T \in \mathcal{L}(V, W) \).
6 Make $P_2(\mathbb{R})$ into an inner product space by defining
\[ \langle p, q \rangle = \int_0^1 p(x)q(x) \, dx. \]

Define $T \in \mathcal{L}(P_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

(a) Show that $T$ is not self-adjoint.

(b) The matrix of $T$ with respect to the basis $(1, x, x^2)$ is
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

This matrix equals its conjugate transpose, even though $T$ is not self-adjoint. Explain why this is not a contradiction.

7 Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that $ST$ is self-adjoint if and only if $ST = TS$.

8 Suppose $V$ is a real inner product space. Show that the set of self-adjoint operators on $V$ is a subspace of $\mathcal{L}(V)$.

9 Suppose $V$ is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on $V$ is not a subspace of $\mathcal{L}(V)$.

10 Suppose $\dim V \geq 2$. Show that the set of normal operators on $V$ is not a subspace of $\mathcal{L}(V)$.

11 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace $U$ of $V$ such that $P = P_U$ if and only if $P$ is self-adjoint.

12 Suppose that $T$ is a normal operator on $V$ and that 3 and 4 are eigenvalues of $T$. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

13 Give an example of an operator $T \in \mathcal{L}(C^4)$ such that $T$ is normal but not self-adjoint.

14 Suppose $T$ is a normal operator on $V$. Suppose also that $v, w \in V$ satisfy the equations
\[ \|v\| = \|w\| = 2, \quad Tv = 3v, \quad Tw = 4w. \]

Show that $\|T(v + w)\| = 10$. 
Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by

$$T v = \langle v, u \rangle x$$

for every $v \in V$.

(a) Suppose $\mathbf{F} = \mathbb{R}$. Prove that $T$ is self-adjoint if and only if $u, x$ is linearly dependent.

(b) Prove that $T$ is normal if and only if $u, x$ is linearly dependent.

Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\text{range } T = \text{range } T^*.$$

Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\text{null } T^k = \text{null } T \quad \text{and} \quad \text{range } T^k = \text{range } T$$

for every positive integer $k$.

Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of $V$ such that $\|T e_j\| = \|T^* e_j\|$ for each $j$, then $T$ is normal.

Suppose $T \in \mathcal{L}(C^3)$ is normal and $T(1, 1, 1) = (2, 2, 2)$. Suppose $(z_1, z_2, z_3) \in \text{null } T$. Prove that $z_1 + z_2 + z_3 = 0$.

Suppose $T \in \mathcal{L}(V, W)$ and $\mathbf{F} = \mathbb{R}$. Let $\Phi_V$ be the isomorphism from $V$ onto the dual space $V'$ given by Exercise 17 in Section 6.B, and let $\Phi_W$ be the corresponding isomorphism from $W$ onto $W'$. Show that if $\Phi_V$ and $\Phi_W$ are used to identify $V$ and $W$ with $V'$ and $W'$, then $T^*$ is identified with the dual map $T'$. More precisely, show that $\Phi_V \circ T^* = T' \circ \Phi_W$.

Fix a positive integer $n$. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx,$$

let

$$V = \text{span}(1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx).$$

(a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that $D$ is normal but not self-adjoint.

(b) Define $T \in \mathcal{L}(V)$ by $Tf = f''$. Show that $T$ is self-adjoint.
Fix \( u, x \in V \). Define \( T \in \mathcal{L}(V) \) by
\[
Tv = \langle v, u \rangle x
\]
for every \( v \in V \).

(a) Suppose \( F = \mathbb{R} \). Prove that \( T \) is self-adjoint if and only if \( u, x \) is linearly dependent.

(b) Prove that \( T \) is normal if and only if \( u, x \) is linearly dependent.

Suppose \( T \in \mathcal{L}(V) \) is normal. Prove that
\[
\text{range } T = \text{range } T^*.
\]

Suppose \( T \in \mathcal{L}(V) \) is normal. Prove that
\[
\text{null } T^k = \text{null } T \quad \text{and} \quad \text{range } T^k = \text{range } T
\]
for every positive integer \( k \).

Prove or give a counterexample: If \( T \in \mathcal{L}(V) \) and there exists an orthonormal basis \( e_1, \ldots, e_n \) of \( V \) such that \( \|Te_j\| = \|T^*e_j\| \) for each \( j \), then \( T \) is normal.

Suppose \( T \in \mathcal{L}(C^3) \) is normal and \( T(1, 1, 1) = (2, 2, 2) \). Suppose \( (z_1, z_2, z_3) \in \text{null } T \). Prove that \( z_1 + z_2 + z_3 = 0 \).

Suppose \( T \in \mathcal{L}(V, W) \) and \( F = \mathbb{R} \). Let \( \Phi_V \) be the isomorphism from \( V \) onto the dual space \( V' \) given by Exercise 17 in Section 6.B, and let \( \Phi_W \) be the corresponding isomorphism from \( W \) onto \( W' \). Show that if \( \Phi_V \) and \( \Phi_W \) are used to identify \( V \) and \( W \) with \( V' \) and \( W' \), then \( T^* \) is identified with the dual map \( T' \). More precisely, show that \( \Phi_V \circ T^* = T' \circ \Phi_W \).

Fix a positive integer \( n \). In the inner product space of continuous real-valued functions on \( [-\pi, \pi] \) with inner product
\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx,
\]
let
\[
V = \text{span}(1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx).
\]

(a) Define \( D \in \mathcal{L}(V) \) by \( Df = f' \). Show that \( D^* = -D \). Conclude that \( D \) is normal but not self-adjoint.

(b) Define \( T \in \mathcal{L}(V) \) by \( Tf = f'' \). Show that \( T \) is self-adjoint.
Due to grader

Dec 11

Relevant to Finals.

Reminder: It works in your favor to participate fully in evaluations of this course.
EXERCISES 7.B

1. True or false (and give a proof of your answer): There exists \( T \in \mathcal{L}(\mathbb{R}^3) \) such that \( T \) is not self-adjoint (with respect to the usual inner product) and such that there is a basis of \( \mathbb{R}^3 \) consisting of eigenvectors of \( T \).

2. Suppose that \( T \) is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of \( T \). Prove that \( T^2 - 5T + 6I = 0 \).

3. Give an example of an operator \( T \in \mathcal{L}(\mathbb{C}^3) \) such that 2 and 3 are the only eigenvalues of \( T \) and \( T^2 - 5T + 6I \neq 0 \).

4. Suppose \( \mathbb{F} = \mathbb{C} \) and \( T \in \mathcal{L}(V) \). Prove that \( T \) is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of \( T \) are orthogonal and

\[
V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),
\]

where \( \lambda_1, \ldots, \lambda_m \) denote the distinct eigenvalues of \( T \).

5. Suppose \( \mathbb{F} = \mathbb{R} \) and \( T \in \mathcal{L}(V) \). Prove that \( T \) is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of \( T \) are orthogonal and

\[
V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),
\]

where \( \lambda_1, \ldots, \lambda_m \) denote the distinct eigenvalues of \( T \).

6. Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

[The exercise above strengthens the analogy (for normal operators) between self-adjoint operators and real numbers.]

7. Suppose \( V \) is a complex inner product space and \( T \in \mathcal{L}(V) \) is a normal operator such that \( T^9 = T^8 \). Prove that \( T \) is self-adjoint and \( T^2 = T \).

8. Give an example of an operator \( T \) on a complex vector space such that \( T^9 = T^8 \) but \( T^2 \neq T \).

9. Suppose \( V \) is a complex inner product space. Prove that every normal operator on \( V \) has a square root. (An operator \( S \in \mathcal{L}(V) \) is called a square root of \( T \in \mathcal{L}(V) \) if \( S^2 = T \).)
The previous result shows that every isometry is normal [see (a), (e), and (f) of 7.42]. Thus characterizations of normal operators can be used to give descriptions of isometries. We do this in the next result in the complex case and in Chapter 9 in the real case (see 9.36).

7.43 Description of isometries when \( F = \mathbb{C} \)

Suppose \( V \) is a complex inner product space and \( S \in \mathcal{L}(V) \). Then the following are equivalent:

(a) \( S \) is an isometry.

(b) There is an orthonormal basis of \( V \) consisting of eigenvectors of \( S \) whose corresponding eigenvalues all have absolute value 1.

Proof We have already shown (see Example 7.38) that (b) implies (a).

To prove the other direction, suppose (a) holds, so \( S \) is an isometry. By the Complex Spectral Theorem (7.24), there is an orthonormal basis \( e_1, \ldots, e_n \) of \( V \) consisting of eigenvectors of \( S \). For \( j \in \{1, \ldots, n\} \), let \( \lambda_j \) be the eigenvalue corresponding to \( e_j \). Then

\[
|\lambda_j| = \|\lambda_j e_j\| = \|Se_j\| = \|e_j\| = 1.
\]

Thus each eigenvalue of \( S \) has absolute value 1, completing the proof.

EXERCISES 7.C

1 Prove or give a counterexample: If \( T \in \mathcal{L}(V) \) is self-adjoint and there exists an orthonormal basis \( e_1, \ldots, e_n \) of \( V \) such that \( \langle Te_j, e_j \rangle \geq 0 \) for each \( j \), then \( T \) is a positive operator.

2 Suppose \( T \) is a positive operator on \( V \). Suppose \( v, w \in V \) are such that

\( Tv = w \) and \( Tw = v \).

Prove that \( v = w \).

3 Suppose \( T \) is a positive operator on \( V \) and \( U \) is a subspace of \( V \) invariant under \( T \). Prove that \( T|_U \in \mathcal{L}(U) \) is a positive operator on \( U \).

4 Suppose \( T \in \mathcal{L}(V, W) \). Prove that \( T^*T \) is a positive operator on \( V \) and \( TT^* \) is a positive operator on \( W \).
5. Prove that the sum of two positive operators on $V$ is positive.

6. Suppose $T \in \mathcal{L}(V)$ is positive. Prove that $T^k$ is positive for every positive integer $k$.

7. Suppose $T$ is a positive operator on $V$. Prove that $T$ is invertible if and only if

$$\langle Tv, v \rangle > 0$$

for every $v \in V$ with $v \neq 0$.

8. Suppose $T \in \mathcal{L}(V)$. For $u, v \in V$, define $\langle u, v \rangle_T$ by

$$\langle u, v \rangle_T = \langle Tu, v \rangle.$$

Prove that $\langle \cdot, \cdot \rangle_T$ is an inner product on $V$ if and only if $T$ is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot \rangle$).

9. Prove or disprove: the identity operator on $\mathbb{F}^2$ has infinitely many self-adjoint square roots.

10. Suppose $S \in \mathcal{L}(V)$. Prove that the following are equivalent:

   (a) $S$ is an isometry;
   (b) $\langle S^*u, S^*v \rangle = \langle u, v \rangle$ for all $u, v \in V$;
   (c) $S^*e_1, \ldots, S^*e_m$ is an orthonormal list for every orthonormal list of vectors $e_1, \ldots, e_m$ in $V$;
   (d) $S^*e_1, \ldots, S^*e_n$ is an orthonormal basis for some orthonormal basis $e_1, \ldots, e_n$ of $V$.

11. Suppose $T_1, T_2$ are normal operators on $\mathcal{L}(\mathbb{F}^3)$ and both operators have $2, 5, 7$ as eigenvalues. Prove that there exists an isometry $S \in \mathcal{L}(\mathbb{F}^3)$ such that $T_1 = S^*T_2S$.

12. Give an example of two self-adjoint operators $T_1, T_2 \in \mathcal{L}(\mathbb{F}^4)$ such that the eigenvalues of both operators are $2, 5, 7$ but there does not exist an isometry $S \in \mathcal{L}(\mathbb{F}^4)$ such that $T_1 = S^*T_2S$. Be sure to explain why there is no isometry with the required property.

13. Prove or give a counterexample: if $S \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of $V$ such that $\|Se_j\| = 1$ for each $e_j$, then $S$ is an isometry.

14. Let $T$ be the second derivative operator in Exercise 21 in Section 7.A. Show that $-T$ is a positive operator.
3. Suppose $T \in \mathcal{L}(V)$. Prove that there exists an isometry $S \in \mathcal{L}(V)$ such that

$$T = \sqrt{TT^*} S.$$ 

4. Suppose $T \in \mathcal{L}(V)$ and $s$ is a singular value of $T$. Prove that there exists a vector $v \in V$ such that $\|v\| = 1$ and $\|Tv\| = s$.

5. Suppose $T \in \mathcal{L}(\mathbb{C}^2)$ is defined by $T(x, y) = (-4y, x)$. Find the singular values of $T$.

6. Find the singular values of the differentiation operator $D \in \mathcal{P}(\mathbb{R}^2)$ defined by $Dp = p'$, where the inner product on $\mathcal{P}(\mathbb{R}^2)$ is as in Example 6.33.

7. Define $T \in \mathcal{L}(\mathbb{F}^3)$ by

$$T(z_1, z_2, z_3) = (z_3, 2z_1, 3z_2).$$

Find (explicitly) an isometry $S \in \mathcal{L}(\mathbb{F}^3)$ such that $T = S \sqrt{T^*T}$.

8. Suppose $T \in \mathcal{L}(V)$, $S \in \mathcal{L}(V)$ is an isometry, and $R \in \mathcal{L}(V)$ is a positive operator such that $T = SR$. Prove that $R = \sqrt{T^*T}$.

[The exercise above shows that if we write $T$ as the product of an isometry and a positive operator (as in the Polar Decomposition 7.45), then the positive operator equals $\sqrt{T^*T}$.]

9. Suppose $T \in \mathcal{L}(V)$. Prove that $T$ is invertible if and only if there exists a unique isometry $S \in \mathcal{L}(V)$ such that $T = S \sqrt{T^*T}$.

10. Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of $T$ equal the absolute values of the eigenvalues of $T$, repeated appropriately.

11. Suppose $T \in \mathcal{L}(V)$. Prove that $T$ and $T^*$ have the same singular values.

12. Prove or give a counterexample: if $T \in \mathcal{L}(V)$, then the singular values of $T^2$ equal the squares of the singular values of $T$.

13. Suppose $T \in \mathcal{L}(V)$. Prove that $T$ is invertible if and only if 0 is not a singular value of $T$.

14. Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of $T$.

15. Suppose $S \in \mathcal{L}(V)$. Prove that $S$ is an isometry if and only if all the singular values of $S$ equal 1.
16. Suppose $T_1, T_2 \in \mathcal{L}(V)$. Prove that $T_1$ and $T_2$ have the same singular values if and only if there exist isometries $S_1, S_2 \in \mathcal{L}(V)$ such that $T_1 = S_1 T_2 S_2$.

17. Suppose $T \in \mathcal{L}(V)$ has singular value decomposition given by

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of $T$ and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of $V$.

(a) Prove that if $v \in V$, then

$$T^* v = s_1 \langle v, f_1 \rangle e_1 + \cdots + s_n \langle v, f_n \rangle e_n.$$  

(b) Prove that if $v \in V$, then

$$T^* T v = s_1^2 \langle v, e_1 \rangle e_1 + \cdots + s_n^2 \langle v, e_n \rangle e_n.$$  

(c) Prove that if $v \in V$, then

$$\sqrt{T^* T} v = s_1 \langle v, e_1 \rangle e_1 + \cdots + s_n \langle v, e_n \rangle e_n.$$  

(d) Suppose $T$ is invertible. Prove that if $v \in V$, then

$$T^{-1} v = \frac{\langle v, f_1 \rangle e_1}{s_1} + \cdots + \frac{\langle v, f_n \rangle e_n}{s_n}$$

for every $v \in V$.

18. Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denote the smallest singular value of $T$, and let $s$ denote the largest singular value of $T$.

(a) Prove that $\hat{s} \|v\| \leq \|Tv\| \leq s \|v\|$ for every $v \in V$.

(b) Suppose $\lambda$ is an eigenvalue of $T$. Prove that $\hat{s} \leq |\lambda| \leq s$.

19. Suppose $T \in \mathcal{L}(V)$. Show that $T$ is uniformly continuous with respect to the metric $d$ on $V$ defined by $d(u, v) = \|u - v\|$.

20. Suppose $S, T \in \mathcal{L}(V)$. Let $s$ denote the largest singular value of $S$, let $t$ denote the largest singular value of $T$, and let $r$ denote the largest singular value of $S + T$. Prove that $r \leq s + t$. 