Math Analysis 2 hwk
Spring 2018 Prof. Lim.
Homework Rules: Tests 50%, Final 30%, Oral HWk Presentations 20%.

(1) Weekly homework sets of 4 to 5 probs from textbook, assigned on Monday.

(2) HWk probs will appear in related questions on the 5-hour tests (one for each chapter, study with Crp. 6)
- roughly 2-3 weeks of hwk for each test.

(3) Tests are closed-book, and will be held on Thursdays, with in-depth test reviews on Monday before tests.

(4) HWk solns/test review are oral presentations by students scheduled every Thursdays that are not test days.

(5) Students will be called alphabetically by last name (and in reverse order) to present prob solns on the board; notes are allowed for these one-hour sessions. Each fully successful hwk soln is worth 2 pts. No shows will be recorded as 0 pts. toward the 20 pts.

(6) Online soln manuals should be consulted only after working hard on the assignments.
EXERCISES

1. Suppose \( x \) increases on \([a, b]\), \( a \leq x_0 \leq b \), \( x \) is continuous at \( x_0 \), \( f(x_0) = 1 \), and 
\( f(x) = 0 \) if \( x \neq x_0 \). Prove that \( f \in \mathcal{R}(x) \) and that 
\( \int_a^b f(x) \, dx = 0 \).

2. Suppose \( f \geq 0 \), \( f \) is continuous on \([a, b]\), and 
\( \int_a^b f(x) \, dx = 0 \). Prove that \( f(x) = 0 \) 
for all \( x \in [a, b] \). (Compare this with Exercise 1.)

3. Define three functions \( \beta_1, \beta_2, \beta_3 \) as follows: \( \beta_j(x) = 0 \) if \( x < 0 \), \( \beta_j(x) = 1 \) if \( x > 0 \) 
for \( j = 1, 2, 3 \); and \( \beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = \frac{1}{2} \). Let \( f \) be a bounded function on 
\([-1, 1]\).

(a) Prove that \( f \in \mathcal{R}(\beta_1) \) if and only if \( f(0+) = f(0) \) and then 
\[ \int f \, d\beta_1 = f(0). \]

(b) State and prove a similar result for \( \beta_2 \).

(c) Prove that \( f \in \mathcal{R}(\beta_3) \) if and only if \( f \) is continuous at 0.

(d) If \( f \) is continuous at 0 prove that 
\[ \int f \, d\beta_1 = \int f \, d\beta_2 = \int f \, d\beta_3 = f(0). \]

4. If \( f(x) = 0 \) for all irrational \( x \), \( f(x) = 1 \) for all rational \( x \), prove that \( f \notin \mathcal{R} \) on \([a, b]\) 
for any \( a < b \).

5. Suppose \( f \) is a bounded real function on \([a, b]\), and \( f^2 \in \mathcal{R} \) on \([a, b]\). Does it 
follow that \( f \in \mathcal{R} \)? Does the answer change if we assume that \( f^2 \in \mathcal{R} \)?

6. Let \( P \) be the Cantor set constructed in Sec. 2.44. Let \( f \) be a bounded real function 
on \([0, 1]\) which is continuous at every point outside \( P \). Prove that \( f \in \mathcal{R} \) on \([0, 1]\).

Hint: \( P \) can be covered by finitely many segments whose total length can be made 
as small as desired. Proceed as in Theorem 6.10.

7. Suppose \( f \) is a real function on \([0, 1]\) and \( f \in \mathcal{R} \) on \([c, 1]\) for every \( c > 0 \). Define 
\[ \int_c^1 f(x) \, dx = \lim_{\epsilon \to 0} \int_c^1 f(x) \, dx \]
if this limit exists (and is finite).

(a) If \( f \in \mathcal{R} \) on \([0, 1]\), show that this definition of the integral agrees with the old 
one.

(b) Construct a function \( f \) such that the above limit exists, although it fails to exist 
with \(|f|\) in place of \( f \).

8. Suppose \( f \in \mathcal{R} \) on \([a, b]\) for every \( b > a \) where \( a \) is fixed. Define 
\[ \int_a^\infty f(x) \, dx = \lim_{\epsilon \to 0} \int_a^\epsilon f(x) \, dx \]
if this limit exists (and is finite). In that case, we say that the integral on the left 
converges. If it also converges after \( f \) has been replaced by \(|f|\), it is said to 
converge absolutely.
Assume that \( f(x) \geq 0 \) and that \( f \) decreases monotonically on \([1, \infty)\). Prove that

\[
\int_{1}^{\infty} f(x) \, dx
\]
converges if and only if

\[
\sum_{n=1}^{\infty} f(n)
\]
converges. (This is the so-called "integral test" for convergence of series.)

9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance, show that

\[
\int_{0}^{\infty} \cos x \, dx = \int_{0}^{\infty} \frac{\sin x}{1 + x^2} \, dx.
\]

Show that one of these integrals converges absolutely, but that the other does not.

10. Let \( p \) and \( q \) be positive real numbers such that

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

Prove the following statements.

(a) If \( u \geq 0 \) and \( v \geq 0 \), then

\[
uv \leq \frac{u^p}{p} + \frac{v^q}{q}.
\]

Equality holds if and only if \( u^p = v^q \).

(b) If \( f \in \mathcal{R}(a), g \in \mathcal{R}(a), f \geq 0, g \geq 0 \), and

\[
\int_{a}^{b} f^p \, dx = 1 = \int_{a}^{b} g^q \, dx,
\]

then

\[
\int_{a}^{b} fg \, dx \leq 1.
\]

(c) If \( f \) and \( g \) are complex functions in \( \mathcal{R}(a) \), then

\[
\left| \int_{a}^{b} fg \, dx \right| \leq \left\{ \int_{a}^{b} |f|^p \, dx \right\}^{1/p} \left\{ \int_{a}^{b} |g|^q \, dx \right\}^{1/q}.
\]

This is Hölder’s inequality. When \( p = q = 2 \) it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder’s inequality is also true for the "improper" integrals described in Exercises 7 and 8.
Hwk 2  Math Analysis 2

Relevant to Test 1 Feb 1 For oral presentations

Mon, Thurs next 2 weeks.

1) Show directly that \( f(x) = x^2, \, \alpha(x) = x \) is \( R(x) \) on \([0,1]\). Your proof should be based on constructing a partition so that \( U - L < \varepsilon \), for any \( \varepsilon > 0 \).

2) Do the same for \( f(x) = x, \, \alpha(x) = x^2 \) on \([0,1]\).

3) For \( f(x) = x^3 \) and \( \alpha(x) = x^3 \), on \([0,1]\) show that the sequence \( \{V_n\} \) where

\[
V_n = U(f, \alpha, \text{partition } (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}))
\]

converges as \( n \to \infty \), to a finite limit \( V \).

4) Estimate \( V \) by calculating \( V_3, V_4, V_{10} \).

5) Calculate \( V \) in (3) exactly.
Hwk #3  For 5011 next Thurs and following Mon.

1. Let \( \alpha(x) \) be an increasing continuous fn. on \([a,b]\), and \( R(\alpha) \)
   is the set of all \( f(x) \) on \([a,b]\) which are Riemann-Stieltjes
   integrable on \([a, b]\).

2. Show that for the above fixed \( \alpha(x) \) and \([a,b]\), \( R(\alpha) \)
   is a real vector space.

3. Let \( W = \{ f \in R(\alpha) \mid \int_a^b \alpha f \, dx = 0 \} \). Show that \( W \subseteq R(\alpha) \)
   is a vector subspace.

4. Show that the "inner product" \( <f+g, h+W> \equiv <f, g> + \int_a^b \alpha \, dh \)
   is well-defined

5. on Quotient space \( R(\alpha)/W \), i.e. show that \( <f+g, h+W> \)
   does not depend on \( f, g \) in the respective cosets \( f+W \), \( g+W \).

6. Check that this "inner product" on \( R(\alpha)/W \) satisfies
   all the properties of a proper inner product.