Trapped slender vortex filaments in statistical equilibrium

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Dedicated to A.

Systems of nearly parallel, slender vortex filaments in which angular momentum is conserved are an important simplification of the Navier-Stokes equations where turbulence can be studied in statistical equilibrium. We study the canonical Gibbs distribution based on the Klein-Majda-Damodaran (KMD) (Klein et al. [1995]) model and find a divergence in the mean square vortex position from that of the point vortex model of Onsager [1949] at high temperature.

1 Introduction

In statistical equilibrium, the behavior of collections of nearly parallel slender vortex filaments, periodic in the z-direction and confined by angular momentum in the xy-directions, is understood to be nearly identical to that of point vortices (in which no z-direction exists) of Onsager [1949] for a wide range of temperatures[1]. When the local self-induction of the filaments, allowing them to vary slightly along their lengths in Brownian motion, is small, it has only a small effect on the macroscopic statistical properties of the system such as density and shape. Taking advantage of this fact, Lions and Majda [2000] were able to rigorously derive a mean-field theory for the density of a system of N nearly parallel vortex filaments using the asymptotically derived PDE of Klein-Majda-Damodaran (KMD) (Klein et al. [1995],), adding mathematical rigor to the statistical equilibrium models for turbulence that Chorin has pioneered Chorin [1994]. However, no adequate theory yet exists for the behavior of the KMD system at high temperature where turbulent behavior is more extreme and long-range order, minimal. Therefore, we must rely on computational methods to explore this important regime. Using the Path Integral Monte Carlo method of Ceperley [1995], originally created to study quantum bosons, we sample the Gibbs canonical ensemble for the Hamiltonian that Klein et al. [1995],. have derived. Our findings show that the point vortex analogy indeed breaks down at high temperatures. While the density of point vortex systems increases monotonically with increasing temperature, we show that the density of the three dimensional system actually *decreases* at some critical temperature and that this entropic behavior occurs at temperatures well below the temperature where the KMD model becomes invalid.

2 Model

2.1 Slender Vortex Filaments

A system of N Nonlinear Schroedinger Equations (NLSEs) describes the time-evolution of slender vortex filaments

$$-i\partial_t \psi_j = \alpha \partial_{\sigma\sigma}\psi_j + \frac{1}{2}\sum_{j\neq k} \frac{\psi_j - \psi_k}{|\psi_j - \psi_k|^2},$$

where $\psi_j(\sigma, t) = x_j(\sigma, t) + iy_j(\sigma, t)$ is the position of vortex j at position σ along its length at time t, and α is the core structure constant (Klein et al. [1995]). Vortex strengths are assumed to all be the same and are set to unity. The position in the complex plane, $\psi_j(\sigma, t)$, is assumed to be periodic in σ with period L.

This system of PDEs can be expressed as a Hamiltonian system

$$H_N = \alpha \int_0^L \sum_{k=1}^N \frac{1}{2}\left|\frac{\partial \psi_k(\sigma)}{\partial \sigma}\right|^2 d\sigma - \int_0^L \sum_{k=1}^N \sum_{i>k} \log|\psi_i(\sigma) - \psi_k(\sigma)| d\sigma.$$

To this we also add a trapping potential which conserves angular momentum,

$$I_N = \int_0^L \sum_{k=1}^N |\psi_k(\sigma)|^2 d\sigma.$$

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[1] By temperature we refer to the Lagrange multiplier for the energy of the vortex system and not the molecular temperature that a thermometer measures.

For our simulations we assume that the filaments are piecewise linear, divided into an equal number of segments of equal length. This discretization leads to the Hamiltonian,

$$H_N(M) = \alpha \sum_{j=1}^{M} \sum_{k=1}^{N} \frac{1}{2} \frac{|\psi_k(j+1) - \psi_k(j)|^2}{\delta} - \sum_{j=1}^{M} \sum_{k=1}^{N} \sum_{i>k}^{N} \delta \log |\psi_i(j) - \psi_k(j)|,$$

and angular momentum

$$I_N = \sum_{j=1}^{M} \sum_{k=1}^{N} \delta |\psi_k(j)|^2,$$

where δ is the length of each segment and M is the number of segments. For purposes of later discussion, the point where two segments meet is called a "bead" in PIMC terminology.

2.2 Path Integrals and Partition Functions

The path integral method of Feynman, in its imaginary time density matrix format, involves evaluating the Gibbs measure of a set of paths,

$$G_N(M) = \frac{\exp\left(-\beta H_N(M) - \mu I_N(M)\right)}{Z_N(M)},$$

where

$$Z_N(M) = \sum_{allpaths} G_N(M).$$

The Gibbs probability, used in the Klein et al. [1995] model by Lions and Majda [2000], gives a probability for a path, and that allows us to use the path integral method. Originally developed for quantum systems of bosons in imaginary time, PIMC applies to the vortex model perfectly as shown by Nordborg and Blatter [1998].

Although we use most of Ceperley's original PIMC methods, there is at least one major difference in notation. In quantum path integral computations β becomes the imaginary time length of the path. Since we are modeling real filaments with their own periodic length, L, it is important to understand that our β corresponds to $1/\hbar$ and not time.

3 Numerical Results

The rest of this paper describes our numerical results based on the model and method described above. Filaments were initialized by scattering them with uniform randomness in a square of side 10, a distance of no particular significance. Monte Carlo moves involved choosing first a vortex filament to change with uniform randomness, then choosing the type of move, either moving the entire chain or rearranging the internal configuration via bisection. If moving the entire chain a new point was chosen within a square of side length one, centered on at the current filament's xy-planar position.

Energy was calculated the same way for both types of moves, using the multilevel method of Ceperley [1995]. Thus, even the wholechain move had the possibility of being rejected before energy was fully evaluated.

We ran our monte carlo simulations until energy settled down to a steady mean. For the high β (i.e. low temperature) simulations we focus on, we see triangular lattices form upon convergence. Although the fluid remains in a liquid state, these lattices have a crystaline structure resembling that of a solid in which the filaments vibrate but maintain a fixed position w.r.t. their neighbors (Figure 1 right.)

Our first finding is a confirmation of findings in Lim and Assad [2005] which demonstrate the relationship between square containment radius, $R = \langle \max_i(|\psi_i|^2) \rangle$, and the parameters β and μ and the circulation Ω, where α was allowed to remain fixed. We find near perfect agreement in the line slopes to Lim and Assad [2005] formula for square containment radius $R^2 = N\beta/(4\pi\mu)$. Additionally, the Hamiltonian used in Lim and Assad [2005] is divided by π, which is not done in Klein et al. [1995]. Therefore, the formula for our square containment radius is

$$R^2 = N\beta/(4\mu),$$

which gives the slope equal to $\beta/(4\mu)$ in agreement Figure 1 left.

Our next finding, shown in Figure 2 is much more important as it deviates from the formula of Lim and Assad [2005]. Using the point vortex analogy, we can get an idea of what is going on. The complete Hamiltonian where the vortices are perfectly straight looks like:

$$H_{straight} = L \left[-\sum_{i=1}^{N} \sum_{k \neq i}^{N} \log |\psi_i(1) - \psi_k(1)| + \frac{\mu}{\beta} \sum_{k=1}^{N} |\psi_k(1)|^2 \right],$$

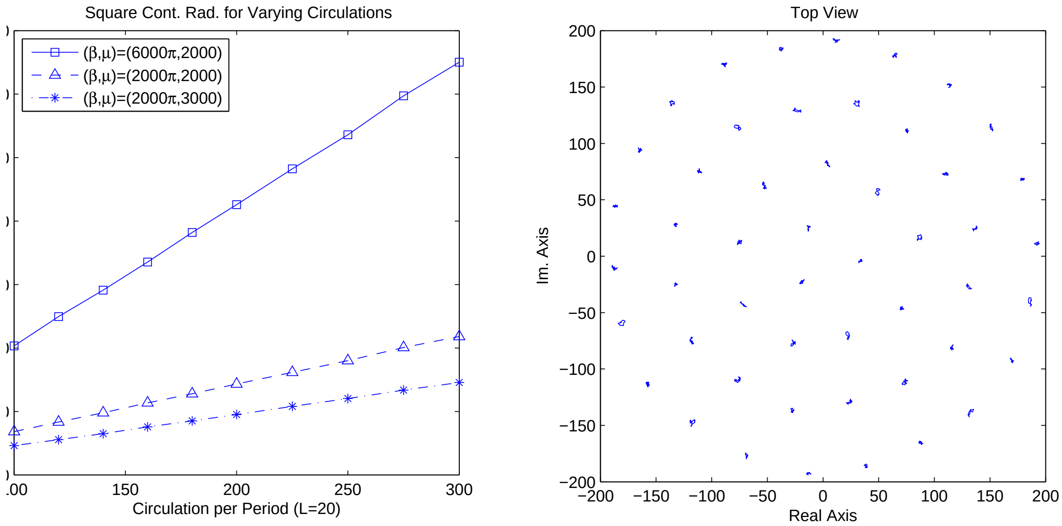

Fig. 1 Low-temperture ensures a well-defined containment radius.

while the Hamiltonian where the vortex planes are independent (no self-induction) looks like:

$$H_{indep} = \sum_{j=1}^{M} \frac{L}{M} \left[- \sum_{k=1}^{N} \sum_{i \neq k}^{N} \log |\psi_i(j) - \psi_k(j)| + \frac{\mu}{\beta} \sum_{k=1}^{N} |\psi_k(j)|^2 \right] .$$

In a per vortex point of view, these two Hamiltonians differ by the constant factor $1/M$. When vortices are straight, they tend to behave like point vortices with vorticity $\sqrt{L}$, whereas, when they are decoupled, they behave like vortices with vorticity $\sqrt{L/M}$. Because, with low β, vortices are quite straight, they behave more like point vortices of the first kind. At very high β, they behave like M systems of point vortices of the second kind. The "V"-shape (where the slope of the line in Figure 2 changes sign) happens when the filaments are making the transition from first to second kind. Only at that β range does their three dimensional nature affect their bulk-R statistics. Furthermore, while the decoupled planes case is unphysical (being dependent on M), the v-shape is not because the asymptotic assumptions are still valid at that point, namely that,

$$L \gg A^2 \gg \epsilon,$$

where A (Figure 3) is the amplitude of the vortices and ϵ is the core size (here assumed to be small). Therefore, this behavior, we assert, is a testable prediction of fluid behavior.

4 Conclusion

To our knowledge no one has done Monte Carlo simulations of the system proposed in Lions and Majda [2000]. Excellent simulations have been done in cases where boundaries are periodic in all directions such as Nordborg and Blatter [1998] and Sen et al. [2001] using the London energy functional for flux line lattices, which differs from that of Klein et al. [1995] only in that the interaction potential is a modifed Bessel's function (log-like at short distances). However, free boundary conditions with the addition of the conservation of angular momentum make this problem remarkably different and specifically applicable to fluid statistics.

Our findings indicate a special regime of temperature where entropy of the filaments has a profound effect on variance of vortex position, a new result to our knowledge. Lions and Majda [2000] have already suggested applicability of their novel derivations in the area of geophysical and astrophysical convection such as Julien et al. [1996] have modeled. We say that our results are equally applicable. What remains is to develop a full free energy functional that can predict R for the widest range of parameters.

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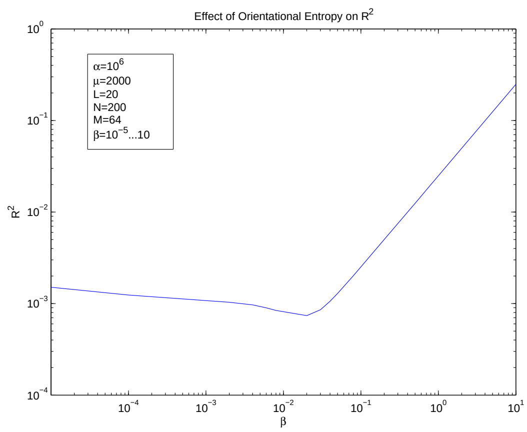

Fig. 2 At a critical β value ($\beta \sim 0.004$) the mean square vortex position begins to increase. Here $L = 20$, $\alpha = 10^6$, $\mu = 2000$, $N = 200$, and $M = 64$.

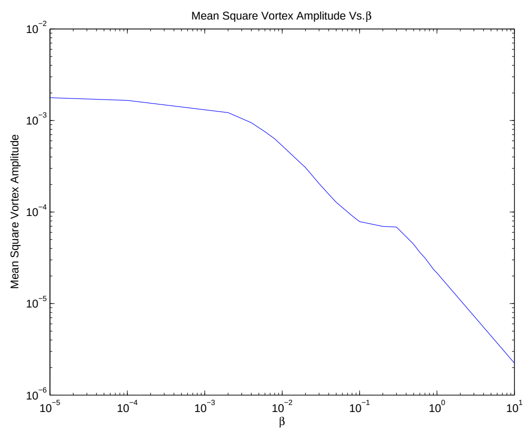

Fig. 3 The amplitude is much less than the length $L = 20$.

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